A Review of General Topology. Part 1: First Concepts

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This document gives the first definitions and results of general topology. It is the first document of a series which reviews the basics of general topology. Other topics such as Hausdorff spaces, metric spaces, connectedness, and compactness will be covered in further documents.

General topology can be viewed as the mathematical study of continuity, connectedness, and related phenomena in a general setting. Such ideas first arise in particular settings such as the real line, Euclidean spaces (including infinite dimensional spaces), and function spaces. An initial goal of general topology is to find an abstract setting that allows us to formulate results concerning continuity and related concepts (convergence, compactness, connectedness, and so forth) that arise in these more concrete settings. The basic definitions of general topology are due to Frèchet and Hausdorff in the early 1900's. General topology is sometimes called point-set topology since it builds directly on set theory. This is in contrast to algebraic topology, which brings in ideas from abstract algebra to define algebraic invariants of spaces, and differential topology which considers topological spaces with additional structure allowing for the notion of differentiability (basic general topology only generalizes the notion of continuous functions, not the notion of differentiable function). We approach general topology here as a common foundation for many subjects in mathematics: more advanced topology, of course, including advanced point-set topology, differential topology, and algebraic topology; but also any part of analysis, differential geometry, Lie theory, and even algebraic and arithmetic geometry.

This document is intentionally concise so it is most suitable for a reader with at least a casual familiarity with topology who is ready to work through a systematic development of some of the key ideas and results of the subject. It is light on counter-examples and skips some less essential topics. Can it be used as a first introduction to general topology? I believe it can if used in conjunction with a knowledgeable instructor or knowlegeable friend, or if supplemented with other less concise sources that discuss examples and motivations. This document might also serve a reader who wishes to review the subject, or as for a quick reference to the basics.

This document gives a rigorous account in the sense that it only relies on results that can be fully proved by the reader without too much trouble given the outlines provided here. The reader is expected to be versed in basic logical and set theoretic techniques employed in the upper-division curriculum of a standard mathematics major. But other than that, the subject is self-contained.[1] I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the prepared reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant. There are several examples included and most of these require the reader to work out various details, so they provide additional exercise.

The approach taken here is standard in general topology, but is quite abstract for a beginner. There is a more concrete setting for much of topology built around the concept of a *metric space*, and one could build a fairly general theory of topology using metric spaces only. General topology partially exists because one finds that in some situations one needs topological notions outside of the context of a metric space. The general approach is taken here, with metric spaces viewed as particular, but important, examples of a topological space. Pedagogically it might be best to start by getting comfortable with topological ideas in the context of Euclidean spaces, or more generally metric spaces. Then one can motivate more general spaces by giving examples – for example manifolds (which are metrizable, but this requires some work, and so are best introduced without a metric), or even non-Hausdorff spaces, such as the Zariski topology of algebraic geometry. At that point the reader would be ready to consider general approach as given in this document. The purpose of this essay is to outline general topology for the reader that is interested and ready for the general approach.

1 Required Background

General topology is a remarkably self-contained subject and does not require much background besides a willingness and ability to think abstractly. General topology is based on set theory so, as mentioned above, elementary set theory is required. This includes the notion of an ordered set. In some examples I assume the notion of a well-ordered set, but more advanced set theoretic notions such as ordinal numbers, cardinal numbers, or forms of the axiom of choice are not required. I do assume the reader is familiar with the field of real numbers $\mathbb{R}$, and in some of the examples I assume a little bit of Euclidean geometry on $\mathbb{R}^2$. I also assume the reader is

[1]Set theoretic reason here is taken to include not just ideas related to intersections, unions, and the empty set, but also complements, functions between arbitrary sets, images and preimages of functions, Cartesian products, relations such as order relations and equivalence relations, well-ordering and so on.

comfortable with mathematical proof and enjoys the challenge of working out proofs for themselves.

2 Topological Structures. Open and Closed Sets.

Topological spaces are a type of mathematical structure. Other mathematical structures include groups, rings, ordered sets, graphs, and so on. Typically structures consist of a set (sometimes multiple sets) equipped with relations or operators for use in the given set (or sets). In topology the set is equipped with a collection of special subsets called *open sets*. From these we define other aspects of the structure such as the collection of closed sets. (There is some degree of arbitrariness in the definition: we could reverse the definition and start with closed subsets, and define open subsets in terms of closed subsets.)

Definition 1 (Topological Space)**.** Let X be a set, which we call the *underlying set*. A *topological structure* on X is a collection $\mathcal{U}$ of subsets called *open subsets* such that the following three "axioms" hold (i) the empty set $\emptyset$ and X as a whole are open subsets, (ii) for any collection $\mathcal{C} \subseteq \mathcal{U}$ of open subsets, the union $\bigcup_{U \in \mathcal{C}} U$ is an open set, and (iii) the intersection of two open sets is an open set.

A *topological space* is a choice of underlying set X together with a choice of topological structure $\mathcal{U}$ on X. Elements $U \in \mathcal{U}$ are called *open subsets* of X.

Remark. A topological structure can be represented as an ordered pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a topological structure on X. However, if in a discussion we consistently use one topological structure on X we can refer to the topological space as X. In other words we can use the same name for the topological space and the underlying space.

Of course, when we are considering two different topological structures on the same underlying set, our notation should distinguish the two topological spaces from each other.

Remark. In this document the term *space* will refer to a topological space. A *topology on* X just means a topological structure on a set X. The term *point* will be used to refer to the elements of the underlying set. If X is the underlying set of a space, then we sometimes use the term *collection* for subsets of the power set of X, and will usually use script notation for such collections (such as $\mathcal{C}$ or $\mathcal{U}$). Officially in set theory this type of collection is just a kind of set, but using different terminology might be helpful in clarifying the concepts. So when we say "subset" or even "set" usually we are referring to a subset of the underlying set, and we use "collection" for sets of such subsets.

Similarly we often use the term *map* for functions between spaces (or technically between the underlying sets of the spaces). Officially a "map" is just a function, but the more geometric language of "map" is often used to go along with the terms "space" and "point".

Remark. The axioms of the above definition can be justified intuitively as follows. Suppose you have some intuitive notion of "space", then you usually have some sort of notion of "neighborhood" of a point x in the space where a "neighborhood" consists of x together with all points "sufficiently close" to x. If your space has a distance, we would want neighborhoods to include at least all points closure than distance ε from x for some $\varepsilon > 0$.

Intuitively, an open subset is a subset U of the space such that for all points $x \in U$ the subset U contains a full neighborhood of x. From this point of view it makes sense to consider the whole space as open since the whole space is a neighborhood (in fact is the largest neighborhood) of each point in the space. The empty set is vacuously open since it has no points. Also, the union of open sets is obviously open according to this notion.

Why is the intersection of two open subsets equal to an open set? Suppose that $x \in U \cap V$ where U and V are open. Then intuitively there is a neighborhood of x contained in U and another neighborhood of x contained in V. So if you believe that the intersection of neighborhoods is a neighborhood, then you would have a neighborhood of x contained in $U \cap V$. For example, if your space happens to have a distance function, then points of distance less than some $\varepsilon_1 > 0$ from x are in U and points of distance less than some $\varepsilon_2 > 0$ are in V. So points of distance less than the minimum of ε_1 and ε_2 would be in $U \cap V$. This supports the idea that the intersection of neighborhood of x should be a neighborhood of x, and so the intersection of open sets is open.

This intuitive justification mentioned the concept of "distance", and distances are usually expressed in terms of real numbers. However, it is important to note that the formal definition does not utilize any concept of distance, and so is independent of the real numbers.

Remark. The requirement that the empty set be open is redundant since technically the empty set is the union $\bigcup_{U \in \mathcal{C}} U$ of the empty collection $\mathcal{C} = \emptyset$. However, I find it friendlier to explicitly assume the empty set is open.

Under the above intuitive notion of open (in terms of neighborhoods), if a point x is isolated from the rest of the set then $\{x\}$ should count as a neighborhood of x. This intuition is reflected in the following definition:

Definition 2. If $\{x\}$ is open then x is called an *isolated point.*

Example 1. If every point of a space X is isolated, then every subset of X must be open. Note that if all subsets of X are declared open then the space X is indeed a topological space since it satisfies all three axioms. We call such a space a *discrete* space since every point is isolated.

At the opposite extreme, if we only declare X and $\emptyset$ to be open then X is also a topological space.[2] Note that if X has at least two elements, this gives a quick example of two distinct topological topological structures on the same topological space.

These first two examples make sense for any underlying set X, and only use set theoretical ideas. Most commonly used topological spaces use geometric or other notions that go beyond just set theory. But there are a few other examples involving basic set theory alone: finite complement topologies (where a nonempty subset is open if and only its complement is finite), countable complement topologies, and so on. We leave it as an exercise to formally define such topologies involving only the concept of cardinality and verify they satisfy the axioms of a topologic space.

[2]This trivial space corresponds to the idea that we have a space where any two points are "arbitrarily close" to each other.

Observe also that our definition requires that even infinite unions of open sets are open. However, we do not usually expect an infinite intersection of open sets to be open. But a finite intersection of open sets will be open:

Proposition 1. *The finite intersection of open sets is open.*

The case of the intersection of two open sets is covered by the definition (requirement (iii) of Definition 1). The general finite case follows, of course, by induction. The statement is valid even for the intersection of zero sets if we define the intersection of zero sets to be the whole space X. In other words, if $\mathcal{C} = \emptyset$ we could define $\bigcap_{U \in \mathcal{C}} U = X$ (using the idea that the intersection of no sets should be "all elements under consideration").

Above we used the intuitive notion of a "neighborhood". Of course any superset of a neighborhood of $x \in X$ should still count as a neighborhood. So, according to the intuitive description of open subsets, any open subset U of X should count as a neighborhood of every $x \in U$. So we make this into a formal definition of "open neighborhood".[3]

Definition 3. Let X be a space and let $x \in X$. An *open neighborhood* of x is an open subset of X containing x.

Intuitively an open subset U is one where every $x \in U$ has a neighborhood contained in U. We make this official, replacing the intuitive notion of *open* neighborhood with that of neighborhood:

Proposition 2. *Let U be a subset of a space X. Then U is open in X if and only if for all $x \in U$ there is an open neighborhood of x contained in U.*

Definition 4. A subset S of a topological space X is *closed* if its complement $X \setminus S$ is open in X.

Proposition 3. *The whole space and the empty set are closed. The set of closed subsets is closed under arbitrary intersection. The finite union of closed subsets is closed.*

3 Bases

It is often convenient to build open sets out of sets of a predetermined form. This leads to the notion of a *basis*.

Definition 5. A *basis* of a topological space X is a collection $\mathcal{B}$ of open subsets of X such that every open set of X is the union of sets from $\mathcal{B}$.

Note that the set of all open sets is obviously a basis. So every topological space has a basis. Usually, however, we like smaller bases made up of special open sets that are particularly easy to define or to work with. We will see several examples

[3] Some authors give a formal definition of a neighborhood that allows non-open sets (including Bourbaki). We won't go down that road here – open neighborhoods are all that we really need and we leave the general notion neighborhood as an intuitive idea.

of this below and in further documents including the collection of open intervals for $\mathbb{R}$ or the collection of open balls in a metric space.

The definition of basis can be rephrased as follows:

Proposition 4. *Let X be a topological space and let $\mathcal{B}$ be a collection of open subsets of X. Then $\mathcal{B}$ is a basis for X if and only if, for all $x \in X$ and all open neighborhoods U of x, there is a B in $\mathcal{B}$ such that $x \in B \subseteq U$.*

There can be many possible natural bases for a given topological space, but each basis uniquely characterizes the topological structure in the following sense:

Proposition 5. *Suppose X_1 and X_2 are two topological spaces with the same underlying set S. Suppose $\mathcal{B}$ is a basis for both X_1 and X_2. Then X_1 and X_2 are the same topological structure on S. In other words, X_1 and X_2 have the same open sets.*

The idea of a basis can be used to *define* a topological structure on a set. Start by nominating a basis and define a subset to be open if and only if it is the union of subsets in the proposed basis. In order to be careful about this process, let's use the term *potential basis* for a collection of subsets that we nominate to be a basis:[4]

Definition 6. A *potential basis* of a set X is a collection $\mathcal{B}$ of subsets of X such that (1) $\bigcup_{U \in \mathcal{B}} U = X$, and (2) given $x \in A \cap B$ where $A, B \in \mathcal{B}$, there is a $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Remark. The condition (1) is often phrased as saying that "$\mathcal{B}$ is a cover of X".

Observe that we do not require that $\mathcal{B}$ be closed under intersection. However, in many examples that follow $\mathcal{B}$ will in fact be closed under intersection. In this case the second condition is automatic. In other words, the following are sufficient conditions for $\mathcal{B}$ to be a potential basis: (1) $\mathcal{B}$ is a cover of X and (2$'$) $\mathcal{B}$ is closed under intersection.

If X is already a topological space then we have the following:

Lemma 6. *Every basis for a topological space is a potential basis.*

We want to be able to use a potential basis to define a topological structure on a set X. The following proposition allows this.

Proposition 7. *Given a potential basis of a set X, there is a unique topological structure on X with the potential basis as a basis. More precisely, define a collection $\mathcal{U}$ of subsets of X by the rule that a subset of X is in $\mathcal{U}$ if and only if it is the union of (zero or more) subsets in the given potential basis. Then $\mathcal{U}$ gives X the structure of a topological space, and the potential basis is a basis for this topology.*

Given a potential basis $\mathcal{B}$ of a set X, the corresponding topological space is said to be the topological space *generated by $\mathcal{B}$*.

[4]The particular term *potential basis* is not standard, but the idea behind it is.

Example 2. In $\mathbb{R}^2$ we can specify two simple potential bases. (i) The collection of open disks, (ii) the collection of interiors of rectangles. Using a few facts from Euclidean geometry we can show that both of these collections are potential bases for $\mathbb{R}^2$. Moreover, using some facts from Euclidean geometry and the following lemma we can show that both of these potential bases define the same topology on $\mathbb{R}^2$.

Lemma 8. *Two potential bases of X generate the same topological structure if and only if (i) every member of the first potential basis is the union of members of the second potential basis, and (ii) vice-versa.*

Remark. We can restate the above by saying that (i) for all U in the first basis, and for all $x \in U$, there is a V in the second basis such that $x \in V \subseteq U$, and (ii) vice-versa.

Remark (Optional). We can extend the idea of *the topology generated by a collection of subsets* from potential bases to arbitrary covers. Let $\mathcal{C}$ be a cover of X. Observe that the collection of all finite intersections of one or more members of $\mathcal{C}$ forms a potential basis since it is closed under intersections. We call this the *potential basis associated with $\mathcal{C}$*.

Let $\mathcal{U}$ be the collection of open sets for the topology generated by the potential basis associated to the cover $\mathcal{C}$. Then (1) every set of $\mathcal{C}$ is open, in other words the inclusion $\mathcal{C} \subseteq \mathcal{U}$ holds, and (2) if $\mathcal{U}'$ is the collection of open sets for some topological structure of X, and if $\mathcal{C} \subseteq \mathcal{U}'$, then $\mathcal{U} \subseteq \mathcal{U}'$. So we can consider $\mathcal{U}$ as the collection of open sets generated by $\mathcal{C}$. Observe that if $\mathcal{C}$ is not a potential basis then $\mathcal{C}$ cannot be a basis of the generated topology. (Munkres calls such a $\mathcal{C}$ a *subbasis*). On the other hand, observe that when $\mathcal{C}$ is already a potential basis, then the potential basis associated with $\mathcal{C}$ (via finite intersections of elements in $\mathcal{C}$) defines the same topology as $\mathcal{C}$ itself.

4 The Order Topology

Now we consider the topology of $X = \mathbb{R}$, or more generally any set X with a given total order. This will give a good example where we can use a basis to define a topology.

We use the term *totally ordered set* (or also *linearly ordered set*) to mean a set X equipped with a total order.[5] Throughout this section let X be a totally ordered

[5]There are various equivalent definitions of a totally ordered set. For example, one can require a relation $<$ with the following properties: (1) $<$ is transitive, (2) given x, y, at least one of the following three holds: $x < y$, $x = y$, $y < x$, and (3) $<$ is anti-reflexive: $\neg(x < x)$ for all $x \in X$. An equivalent definition requires (1) $<$ is transitive, (2′) given x, y, exactly one of of the following three holds: $x < y$, $x = y$, $y < x$.

We call such a relation $<$ a "strict total order". One can also approach the definition of totally ordered set by characterizing $\leq$ instead of $<$. If $<$ is a total order then we define $x \leq y$ as $(x < y) \vee (x = y)$. Then $\leq$ is reflexive, antisymmetric (in the sense that if $x \leq y$ and $y \leq x$ then $x = y$), transitive, and finally has the property that $(x \leq y) \vee (y \leq x)$ for all $x, y \in X$. We call this relation $\leq$ a "reflexive total order" or a "nonstrict total order". Conversely if $\leq$ satisfies these four properties, then we can define $x < y$ as $(x \leq y) \wedge (x \neq y)$, and can show that $<$ is a strict total order in the above sense. Note that if $\leq$ satisfies the first three of these four properties then it is called a "partial order". Note also that reflexivity for $\leq$ follows from the fourth property, so stating it for a reflexive total order is redundant.

set.

For convenience we will fix two distinct elements not in X, which we will call ∞ and $-\infty$. We consider the (disjoint) union $X \cup \{-\infty, \infty\}$ which we call X^*. We extend the order relation to X^* in the obvious way. Then X^* is seen to be a totally ordered set with smallest element $-\infty$ and largest element ∞.

Given $a, b \in X^*$, we define the *open interval* (a, b) in the usual way as the set of $x \in X$ such that $a < x < b$. We allow the empty set to be an open interval; for example, if $a \in X$ then (a, a) is empty. It is easy to verify that open intervals satisfy the simple law

$$(a, b) \cap (c, d) = \big(\max(a, c), \min(b, d)\big).$$

In particular, the intersection of two open intervals is an open interval. Also note that $(-\infty, \infty) = X$. Thus the set of open intervals forms a potential basis for X. The associated topology which has the set of open intervals as a basis is called the *order topology*, and we call X with this collection of open sets an *ordered space*.[6]

For the record we state the following:

Proposition 9. *Let X be an ordered space. Then every open interval of X is open. In fact, the collection of open intervals is a basis for X.*

Example 3. In particular, well-ordered sets are topological spaces using the order topology. Observe that for well-ordered sets, a point is isolated if and only if it is the first point or the immediate successor of another point. (A point x is isolated means that $\{x\}$ is open.)

We define *closed intervals* in the usual way. If $a, b \in X$ then

$$[a, b] \stackrel{\text{def}}{=} \{x \in X \mid a \leq x \leq b\}.$$

We call $[a, b]$ a *closed interval*.

Lemma 10. *Suppose $a, b \in X$, and let $[a, b]^c$ be the complement of $[a, b]$ in X. Then*

$$[a, b]^c = (-\infty, a) \cup (b, \infty).$$

Proposition 11. *Let X be an ordered space. Every closed interval of X is a closed subset of X.*

We define *half open intervals* $(a, b]$ and $[c, d)$ in the usual way. Here we want $b, c \in X$, but we allow infinite values for a and d: so $a, d \in X^*$. In particular, half intervals are (possibly empty) subsets of X. An *interval* is any subset of X that is either an open interval, a closed interval, or a half-open interval. We will use the general notion interval in Lemma 35 below.

5 Closure and Limit Points

Definition 7. Let S be a subset of a space X. The *closure* $\overline{S}$ of S in X is the intersection of all closed subsets of X containing S.

[6] So in this document when we use the term "ordered space" for a space X, we assume that X is a *totally* ordered set and that the collection of open intervals forms a basis.

The closure of S is the smallest closed subset of X containing S:

Proposition 12. *Let S be a subset of a space X. Then $S \subseteq \overline{S}$ and $\overline{S}$ is a closed subset of X. Furthermore, if Z is a closed subset of X with $S \subseteq Z$ then $\overline{S} \subseteq Z$.*

A consequences of this proposition is the following:

Proposition 13. *Let A and B be subsets of a space X. Then*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Proposition 14. *A subset S of X is closed if and only if $S = \overline{S}$.*

The following two theorems give more concrete descriptions of the closure.

Proposition 15. *Let S be a subset of a space X. Then $x \in \overline{S}$ if and only if all open neighborhoods U of x intersect S.*

Proposition 16. *Let X be a space with basis $\mathcal{B}$. Let S be a subset of X. Then $x \in \overline{S}$ if and only if all $B \in \mathcal{B}$ containing x also intersect S.*

We end with the notion of a limit point.

Definition 8. Let S be a subset of a space X. A *limit point* of S is a point x such that every open neighborhood of x intersects S in a point not equal to x.

Proposition 17. *Let X be a space with basis $\mathcal{B}$, and let S be a subset of X. Then x is a limit point of S if and only if, for all neighborhoods B of x in $\mathcal{B}$, the intersection $B \cap S$ contains a point not equal to x.*

Proposition 18. *Let S be subset of a space X, let $\overline{S}$ be the closure of S, and let S' be the set of limit points of S. Then*

$$\overline{S} = S \cup S'.$$

Proposition 19. *A subset of a space X is closed in X if and only if it contains all of its limit points.*

6 Interior Points and Boundaries

Next we consider (i) the notion of *interior* which is a dual to the notion of closure, and (ii) the notion of *boundary*.

Definition 9. Let S be a subset of a space X. The *interior* of S is the union of all open subsets contained in S.

The interior of S is the largest open subset contained in S:

Proposition 20. *Let S be a subset of a space X, and let I be the interior of S. Then $I \subseteq S$ and I is open in X. Furthermore, if W is an open subset with $W \subseteq S$ then $W \subseteq I$.*

Proposition 21. *Let S be a subset of a space X. Then S is open if and only if S equals the interior of S.*

Definition 10. The boundary of a subset S of a space X is defined to be the intersection of $\overline{S}$ and $\overline{X - S}$. In other words, x is in the boundary of S if and only if every open neighborhood of x contains points inside and outside of S.

Proposition 22. *A set and its complement have the same boundary.*

Proposition 23. *The closure $\overline{S}$ is the disjoint union of the interior and the boundary of S. The space X as a whole is the disjoint union of (i) the interior of S, (ii) the boundary of S, (iii) the interior of the complement of S.*

Corollary 24. *Let Z be a closed subset of X. Then Z is the disjoint union of the interior of Z and the boundary of Z.*

Proposition 25. *A subset is "clopen" (open and closed) if and only if it has an empty boundary.*

7 The Subspace Topology

Any subset of a topological space is automatically itself a topological space using the subspace topology:

Definition 11. Let X be a topological space and let Y be a subset of X. Then the *subspace* topology on Y is the topology created by declaring a subset $V \subseteq Y$ to be open in Y if and only if there is an open subsset U of X such that $V = U \cap Y$. We call Y equipped with this topology a *subspace* of X.

Lemma 26. *The above definition defines a topology on Y.*

Proposition 27. *Let Y be a subspace of X. Let Z be a subset of Y. Then Z is closed in Y if and only if there is a closed subset W of X such that $Z = W \cap Y$.*

Proposition 28. *Suppose $\mathcal{B}$ is a basis of a space X. Let Y be a subspace of X. Then the collection of sets of the form $B \cap Y$ with $B \in \mathcal{B}$ forms a basis of Y.*

When using subspaces Y of a space X, we have to be very careful about phrases such as "A is open" or "A is closed" since there is ambiguity. In fact we should regard open and closed as relative notions. We should distinguish "open in X" from "open in Y" or "closed in X" from "closed in Y". The following propositions give situations where these notions align:

Proposition 29. *Let Y be an open subspace of X (a subspace such that the set Y is open in X). Then a subset of Y is open in Y if and only if it is open in X.*

Proposition 30. *Let Y be an closed subspace of X (a subspace such that the set Y is closed in X). Then a subset of Y is closed in Y if and only if it is closed in X.*

There is a kind of "transitive law" for subspaces:

Proposition 31. *Let Y be a subspace of X. Let Z be a subset of Y. The subspace topology of Z considered as a subset of Y is the same as the subspace topology of Z considered as a subset of X.*

Closures are well behaved:

Proposition 32. *Let S be a subset of Y where Y is a subspace of X. Let $\overline{S}$ be the closure of S in X. Then the closure of S in Y is $\overline{S} \cap Y$.*

8 The Subspace Topology for Order Topologies

If S is a subset of an ordered set then S can be made into a topological space using the order relation in two ways. (1) S can be given the subset topology. In addition, (2) S is itself an ordered set (using the induced order) and so has an order topology. Note that these two topologies do not always coincide. For example, consider $\mathbb{R}$ with the order topology and let $S = \mathbb{R} - [0, 1)$. Then $[1, \infty)$ is an open subset of S according to the subset topology. However, it is not open according to the order topology of S.

We do have, however, the following lemma and corollary:

Lemma 33. *Let X be an ordered set and S a subset. Let $a, b \in S^*$, let $(a, b)_X$ the corresponding open interval in X, and let $(a, b)_S$ be the corresponding open interval in S. Then*

$$(a, b)_S = (a, b)_X \cap S.$$

Thus $(a, b)_S$ is open in both the subspace topology and in the order topology of S.

Corollary 34. *Let X be an ordered set, and S a subset. Then every open subset of S according to the order topology of S is also open in the subset topology of S.*[7]

There is an important case where the two topologies on S do coincide. This is the case where S is convex.

Definition 12. A subset S of an ordered set X is said to be *convex* if, for all $x \in X$, if $a \leq x \leq b$ for some $a, b \in S$ then $x \in S$.

Remark. An equivalent characterization of convex is that if $x \notin S$ then x is a lower bound of S or x is a upper bound of S.

Recall the definition of *interval* allows for open intervals, closed intervals, or half-open intervals (see Section 4).

Lemma 35. *Every interval in an ordered space X is convex.*

Remark. The converse of the above holds for certain ordered spaces such as $X = \mathbb{R}$.

[7]Some authors express this by saying that the order topology on S is *finer* than the subset topology, or that the subset topology is *courser* than the order topology.

Lemma 36. *Let S be a convex subset of an ordered set X. If (a, b) is an open interval of X then $(a, b) \cap S$ is an open interval in S according to the order topology of S.*

Proof. If $(a, b) \cap S$ is empty then the result follows immediately (the empty set is considered an open interval), so assume that $(a, b) \cap S$ is nonempty.

If a is a strict lower bound of S then let $a' = -\infty$, otherwise let $a' = a$. In the second case $a \in S$ by convexity. If b is a strict upper bound of S then let $b' = \infty$, otherwise let $b' = b$. In the second case $b \in S$ by convexity. Observe that $a', b' \in S^*$ and that $(a', b') \cap S = (a, b) \cap S$.

Let $(a', b')_S$ be the open interval according to the order relation restricted to S. By Lemma 33,

$$(a', b')_S = (a', b') \cap S = (a, b) \cap S.$$

□

Corollary 37. *Let S be a convex subset of an ordered set X. Then the subset topology of S coincides with the order topology of S.*

Remark. There are other examples where the subset topology aligns with the order topology. For instance $\mathbb{Q}$ considered as an ordered set has the same topology as $\mathbb{Q}$ considered as a subspace of $\mathbb{R}$.

Note that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$, and that the following three topologies on $\mathbb{Z}$ are equivalent: (1) the order topology, (2) the subset topology, and (3) the discrete topology.

9 Continuous Functions

In topology we do not usually consider general functions, but restrict our attention to functions that "respect" the topological structure in some way. These functions include continuous functions, open functions, and homeomorphisms. (A homeomorphism is both continuous and open as we will see). The most important of these types of functions are the continuous functions. In fact one *purpose* of general topology is to set up a notion of continuous functions in a general setting.

Definition 13. Let $f : X \to Y$ be a function between topological spaces. The function f is said to be *continuous* if the preimage of every open subset of Y is open in X.

Remark. Recall that union and intersection commute with preimage. Similarly, complements commutes with preimage. These facts can be used to prove the following two propositions.

Example 4. Any function from a discrete space into any given topological space is continuous.

Proposition 38. *Let $f : X \to Y$ be a function between topological spaces. Suppose we fix a basis for Y. Then the function f is continuous if and only if the preimage of every open set from the basis of Y is open in X.*

Proposition 39. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if the preimage of every closed subset of Y is closed in X.*

The following easy lemma from set theory will be useful:

Lemma 40. *Suppose that $f\colon X \to Y$ is a function, $U \subseteq X$, and $V \subseteq Y$. Let $f|_U$ be the restriction of f to U which we regard as a function $U \to Y$. Then*

$$(f|_U)^{-1}[V] = f^{-1}[V] \cap U.$$

This lemma can be used to prove the following:

Proposition 41. *Suppose $f\colon X \to Y$ is a continuous function between topological spaces. Let Z be a subspace of X. Then the restriction $f|_Z$ is continuous.*

There is another way to think about continuity:

Proposition 42. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if it preserves the "membership in the closure" relation. (This condition can be stated as follows: for all $A \subseteq X$ and $x \in X$, if $x \in \overline{A}$ then $f(x) \in \overline{f[A]}$. Another way to state this is as follows: $f\left[\overline{A}\right] \subseteq \overline{f[A]}$).*

Proof. Assume continuity, and assume that $f(x)$ is not in the closure of $f[A]$. Then there is a an open subset V of Y containing $f(x)$ which is disjoint from $f[A]$. In particular $f^{-1}[V]$ is an open neighborhood of x disjoint from A. So x cannot be in the closure of A.

Conversely, assume f preserves the "membership in the closure" relation. Suppose there is a closed subset Z of Y such that $f^{-1}[Z]$ is not closed. So $f^{-1}[Z]$ has a limit point $x \in X$ not in $f^{-1}[Z]$. By assumption, $f(x)$ is in the closure of the image of $f^{-1}[Z]$. But the image of $f^{-1}[Z]$ is a subset of Z. So $f(x)$ is in the closure of Z. Since Z is closed, $f(x) \in Z$. But x is not in $f^{-1}[Z]$, a contradiction. □

Here are some fundamental results about continuous functions.

Proposition 43. *The composition of continuous functions is continuous.*

Proposition 44. *Suppose X is a topological space and Z is a subspace of X. Then the inclusion map $Z \to X$ is continuous. In particular, the identity map $X \to X$ is continuous.*

Remark. We can use the above two results to give another proof of Proposition 48: just consider $f \circ \iota$ where ι is the inclusion map.

The proof of the following only requires the fact that the empty set and the whole space are open:

Proposition 45. *Constant maps between topological spaces are continuous.*

Given a function $f\colon X \to Y$, we can always replace the codomain Y with a subset Z of Y as long as Z contains the image of f. This map $f'\colon X \to Z$, which differs from f only in what the codomain is designated to be, is called the *restriction of codomain* of f (in particular, $f(x) = f'(x)$ for all $x \in X$). The following shows continuity is well-behaved with respect to this operation.

Proposition 46. *Suppose $f : X \to Y$ is a function between topological spaces whose image is contained in a subspace $Z \subseteq Y$. Then f is continuous if and only if corresponding restriction of codomain $f' : X \to Z$ is continuous.*

10 Open Mapping

In addition to the notion of a continuous function we have the notion of an open mapping:

Definition 14. Let $f : X \to Y$ be a function between topological spaces X and Y. The function is said to be an *open mapping* if $f[U]$ is open in Y for all open subsets U of X.

Proposition 47. *Let $f : X \to Y$ be a function between topological spaces. Suppose we fix a basis $\mathcal{B}$ for X. Then the function f is an open mapping if and only if $f[U]$ is open in Y for all U in $\mathcal{B}$.*

Proposition 48. *Suppose $f : X \to Y$ is an open mapping between topological spaces. Let U be an open subspace of X. Then the restriction $f\mid_U$ is an open mapping.*

Proposition 49. *The composition of open mappings is an open mapping.*

Proposition 50. *Suppose X is a topological space. Then the identity map $X \to X$ is an open mapping.*

There is a connection between continuous functions and open mappings. Before we make this connection we need to discuss an ambiguity of notation. If $f : X \to Y$ is a bijection between two sets and if V is a subset of Y then the notation $f^{-1}[V]$ is ambiguous, it has two interpretations. It can refer to the preimage of V under f, or it can refer to the the image of V under f^{-1}. It turns out that both of these interpretations yield the same set, so we can tolerate the ambiguity. (If f is not a bijection then there is no ambiguity: the notation refers to the preimage.) We state this as a lemma from set theory:

Lemma 51. *Suppose $f : X \to Y$ is a bijection between two sets and V is a subset of Y. Let A be the preimage $f^{-1}[V]$ of V under f. Let B be the image $(f^{-1})[V]$ under f^{-1}. Then $A = B$.*

Now we can give the connection between continuous functions and open mappings in the case of bijective functions:

Proposition 52. *Suppose $f : X \to Y$ is a bijection between topological spaces. Then f is continuous if and only if f^{-1} is an open mapping.*

Remark. This ends our initial discussion of open mappings. We could define the notion of "closed mappings" as well, but we will not have need for such functions in this document. But observe that if f is a bijective open mapping, then it must map closed sets to closed sets.

11 Homeomorphisms

In general, two mathematical structures of a certain kind are "isomorphic" if they are "essentially the same". What does it mean for two topological spaces to be "essentially the same"? In many contexts, two mathematical structures are isomorphic if there is a bijection between the underlying sets such that when you replace each element from one underlying set with the corresponding element of the other underlying set you transform the first mathematical structure to be equal to the second structure.[8]

Since topological structures are defined in terms of open sets, this means that a "topological isomorphism" would be one where a set in the domain is open if and only if its image in the codomain is open. For historical reasons the term "homeomorphism" is used instead of "isomorphism" in the context of topological structures.

Definition 15. Let $f : X \to Y$ be a function between topological spaces. We say that f is a *homeomorphism* if f is a bijection with the following property: for all subsets $U \subseteq X$, the subset U is open in X if and only if its image $f[U]$ is open in Y.

Proposition 53. *Let $f : X \to Y$ be a bijection between topological spaces. Then f is a homeomorphism if and only if f is an open mapping that is continuous.*

Proof. Start by establishing the following set-theoretic law:

$$V = f[U] \iff U = f^{-1}[V]$$

which applies to all subsets $U \subseteq X$ and $V \subseteq Y$. □

Here is a common description of homeomorphisms:

Proposition 54. *Let $f : X \to Y$ be a function between topological spaces. Then f is a homeomorphism if and only if (1) f is bijective, (2) f is continuous, and (3) the inverse f^{-1} is continuous.*

Of course, we could also describe homeomorphisms as follows:

Proposition 55. *Let $f : X \to Y$ be a function between topological spaces. Then f is a homeomorphism if and only if (1) f is bijective, (2) f is an open mapping, and (3) the inverse f^{-1} is an open mapping.*

Here are some basic results about homeomorphism.

Proposition 56. *Suppose X is a topological space. Then the identity $X \to X$ is a homeomorphism.*

Proposition 57. *The inverse of a homeomorphism is a homeomorphism.*

[8]For example, if G and H are groups then an isomorphism $f : G \to H$ is a bijection such that f transforms the binary operation defining G to the binary operation defining H. In other words, we have $z = x \cdot y$ in G if and only if we have $f(z) = f(x) \cdot f(y)$ in H (where here the operation is the operation of H). For a bijection f this is equivalent to requiring that $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in G$.

Proposition 58. *The composition of homeomorphisms is a homeomorphism.*

Remark. Not every continuous bijection is a homeomorphism. A classical example is the function

$$t \mapsto (\cos 2\pi t, \sin 2\pi t) \, .$$

Using a few basic facts from trigonometry, we can show that this is a bijection from the interval $[0, 1)$ in $\mathbb{R}$ to the unit circle. If we give the unit circle the subspace topology inside $\mathbb{R}^2$, and we assume we have already established that sin and cos are continuous, then we can prove that this function is continuous. (Use open rectangles as a basis for $\mathbb{R}^2$. Use the subspace or order topology on $[0, 1)$; we have seen that these two topologies are equivalent.) However, $[0, 1/2)$ is an open subset of $[0, 1)$ which does not map to an open subset of the unit circle.

Remark. Here is another example of a continuous bijection that is not a homeomorphism. It is more self-contained since it avoids trigonometry, but perhaps it is not as natural as the previous example: let X_1 be $\mathbb{R} - [0, 1)$ with the subspace topology, considered as a subspace of $\mathbb{R}$. Let X_2 be the same set but with the order topology. Let $X_1 \to X_2$ be the identity map. Then by Corollary 34 this is a continuous bijection. However, it is not a homeomorphism since $[1, \infty)$ is open in X_1 but not in X_2.

Lemma 59. *Suppose $f \colon X \to Y$ is a bijection between topological spaces. Suppose X has basis $\mathcal{B}_1$ and suppose Y has basis $\mathcal{B}_2$. Finally, suppose that $f[A] \in \mathcal{B}_2$ for all $A \in \mathcal{B}_1$, and $f^{-1}[B] \in \mathcal{B}_1$ for all $B \in \mathcal{B}_2$. Then f is a homeomorphism.*

Proposition 60. *Let $f : X \to Y$ be an order-preserving bijective function between order spaces. Then f^{-1} is order-preserving. The image of the interval is an interval. Furthermore, f is a homeomorphism.*

Remark. This shows, modulo some facts from algebra and calculus, that the function $x \mapsto x/(1 - x^2)$ is a homeomorphism between $(-1, 1)$ and $\mathbb{R}$. Moreover, we can use this to prove that all nonempty open intervals of $\mathbb{R}$ are homeomorphic to each other.

Here is a tool to show certain spaces are homeomorphic. We will use this later when discussing product topologies. It shows that even when we start with "one-sided" invertibility, continuity of both functions is enough to yield a homeomorphism.

Proposition 61. *Suppose that $f \colon X \to Y$ and $g \colon Y \to X$ are continuous maps such that $g \circ f$ is the identity map $X \to X$. Let Y' be the image of f. Then the restricted of codomain map $f' \colon X \to Y'$ is a homeomorphism with inverse $g\,|_{Y'}$.*

Proof. Start by establishing the following set-theoretic identity

$$f[U] = g^{-1}[U] \cap Y'$$

for all subsets U of X. □

Here are some other tools.

Proposition 62. *Suppose that $f: X \to Y$ is a homeomorphism between spaces. Suppose A is a subspace of X, and let the image $B = f[A]$ be considered as a subspace of Y. Then f restricts to a homeomorphism $A \to B$.*

Proposition 63. *Suppose that $f: X \to Y$ is a bijection where X is a topological space and where Y is just a set. Define a set in Y to be "open" if it is the image of an open set of X under f. Then this collection of open sets gives a topological structure to Y, and it is the unique topological structure such that f is a homeomorphism.*

12 Continuity at a Point

Definition 16. Suppose that $f: X \to Y$ is a function between topological spaces. We say that *f is continuous at point $x_0 \in X$* if for all open neighborhoods V of $f(x_0)$ there is an open neighborhood U of x_0 such that $f[U] \subseteq V$.

Note: to check that $f: X \to Y$ is continuous at a point, it is enough to check the above condition for open neighborhoods V of $f(x_0)$ in a basis. In fact, it is enough to check the condition for a neighborhood basis for $f(x_0)$:

Definition 17. Suppose X is a topological space and let $x \in X$ be a point. A *neighborhoood basis for x* is a collection $\mathcal{B}$ of open neighborhoods of $x \in X$ with the following property: if U is an open neighborhood of x then there is an open neighborhood B of x in $\mathcal{B}$ such that $B \subseteq U$.

Using Proposition 4 we get the following:

Lemma 64. *Let X be a topological space with basis $\mathcal{B}$. Let $x \in X$, and let $\mathcal{B}_x$ be the collection of open neighborhoods B of x such that B is in $\mathcal{B}$. Then $\mathcal{B}_x$ is a neighborhood basis for x.*

Often continuity at a point is expressed in terms of neighborhood basis:

Proposition 65. *Let $f: X \to Y$ be a function between topological spaces and let $x_0 \in X$. Let $\mathcal{B}_{x_0}$ be a neighborhood basis of x_0, and let $\mathcal{B}_{f(x_0)}$ be a neighborhood basis of $f(x_0)$. Then f is continuous at x_0 if for all neighborhoods V in $\mathcal{B}_{f(x_0)}$ there is a neighborhood U in $\mathcal{B}_{x_0}$ such that $f[U] \subseteq V$.*

Remark. For example, if $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a function and if $x_0 \in \mathbb{R}^2$ is a given point, then the open disks centered at x_0 forms a neighborhood basis of x_0. Similarly, the open disks centered at $f(x_0)$ forms a neighborhood basis of $f(x_0)$. So by the above proposition, f is continuous at x_0 if and only if for any $\varepsilon > 0$, with associated open disk V_ε of radius ε and center $f(x_0)$, there is a $\delta > 0$, with associated open disk U_δ and center x_0, such that $f[U_\delta] \subseteq V_\epsilon$. A similar characterization applies to functions $\mathbb{R} \to \mathbb{R}$, where open intervals replace open disks. These descriptions corresponds to the tradition definitions of continuity at a point given in an introductory analysis course.

Proposition 66. *Suppose that $f: X \to Y$ is a function from a space X to a space Y. Then f is continuous if and only if it is continuous at x for all $x \in X$.*

Proof. If f is continuous, continuity for each $x \in X$ is straightforward.

Now suppose that f is continuous at each $x \in X$. Let V be an open subset of Y. We will use Proposition 2 to show that $f^{-1}[V]$ is open in X. Let $x \in f^{-1}[V]$, so V is a neighborhood of $f(x)$. By local continuity at x, there is a neighborhood U of x such that $f[U] \subseteq V$. This means $U \subseteq f^{-1}[V]$ as desired. This establishes that $f^{-1}[V]$ is open. □

Continuity at a point is a local concept in the following sense:

Proposition 67. *Let $f : X \to Y$ be a function between topological spaces and let U be an open neighborhood of $x_0 \in X$. Then f is continuous at x_0 if and only if the restriction $f|_U : U \to Y$ is continuous at x_0.*

13 The Product Topology

Given X and Y two topological spaces, we can give a natural topological structure to the Cartesian product $X \times Y$. In a future document we will also describe the topology of infinite Cartesian products, but for now we will restrict to this simpler situation.

In this section, we will make extensive use of the following basic set-theoretical identity for A and Z subsets of a set X, and B and W subsets of a set Y:

$$(A \cap Z) \times (B \cap W) = (A \times B) \cap (Z \times W).$$

In particular, this identity gives a quick proof of the following:

Lemma 68. *Let $X \times Y$ be the product of two topological spaces. The collection of subsets of the form $U \times W$, where U is open in X and W is open in Y, is closed under finite intersections. In particular, this collection is a potential basis.*

Definition 18. The *product topology* on $X \times Y$ is the topology generated by the above potential basis.

We state the following proposition "for the record"; it is an immediate consequence of the definition.

Proposition 69. *Let X and Y be spaces. If U is an open subset of X and if W is an open subset of Y then $U \times W$ is an open subset of $X \times Y$.*

There is a corresponding statement for closed subsets:

Proposition 70. *Let X and Y be spaces. Suppose $A \subseteq X$ and $B \subseteq Y$. If A and B are closed subsets of their respective spaces then $A \times B$ is closed in $X \times Y$. In general, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Here is an easy way to get bases for Cartesian products:

Proposition 71. *Let $X \times Y$ be the product of two topological spaces. Let $\mathcal{B}_1$ be a basis of X and $\mathcal{B}_2$ be a basis of Y. Then the collection of sets of the form $U \times W$ with U in $\mathcal{B}_1$ and W in $\mathcal{B}_2$ is a basis of $X \times Y$.*

Proof. This is straightforward using Proposition 4 and the definition of the product topology. □

When one Cartesian product is a subspace of another Cartesian product, the subset topology is well behaved:

Proposition 72. *Let X and Y be spaces. Let $Z \subseteq X$ and $W \subseteq Y$ be subspaces (with the subspace topologies). Then the subspace topology on $Z \times W$, considered as a subspace of $X \times Y$, is the same as the product topology on $Z \times W$.*

Proof. Let $\mathcal{B}$ be the collection of sets of the form $A \times B$ where A is open in Z and B is open in W. Then we can check that $\mathcal{B}$ is a common basis for the two topological structures in question (using Proposition 4 where necessary). □

Now we consider functions related to Cartesian products.

Proposition 73. *The function $(x, y) \mapsto (y, x)$ defines a homeomorphism*

$$X \times Y \to Y \times X.$$

Proof. See Proposition 38. □

The following shows that the projection maps are well-behaved. (In the proof be careful to distinguish cases where one or both terms of a Cartesian product is empty.)

Proposition 74. *Let $X \times Y$ be a product space of X and Y. Then both projection functions $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are continuous functions that are open mappings.*

Proposition 75. *Let $f \colon Z \to X \times Y$. Let $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ be the corresponding coordinate functions where π_1 and π_2 are the two projection functions. Then f is given by the formula $f(z) = (f_1(z), f_2(z))$. Furthermore, f is continuous if and only if both f_1 and f_2 are continuous.*

Proof. To show continuity of f, first show $f^{-1}[U_1 \times U_2] = f_1^{-1}[U_1] \cap f_2^{-1}[U_2]$. □

Proposition 76. *If $f \colon X \to X'$ and $g \colon Y \to Y'$ are continuous, then so is the function*

$$X \times Y \to X' \times Y'$$

defined by $(x, y) \mapsto (fx, gy)$.

Proposition 77. *Let X and Y be spaces. If $y_0 \in Y$ then the function $x \mapsto (x, y_0)$ defines a homeomorphism between X and the subspace $X \times \{y_0\}$ of $X \times Y$. Similarly, if $x_0 \in X$ then the function $y \mapsto (x_0, y)$ defines a homeomorphism between Y and the subspace $\{x_0\} \times Y$ of $X \times Y$*

Proof. See Proposition 61. □

Proposition 78. *Let X be a space. The function $x \mapsto (x, x)$ defines a homeomorphism between X and the "diagonal' $\Delta = \{(x, x) \mid x \in X\}$ considered as a subspace of $X \times X$.*

Proof. See Proposition 61. □

This generalizes to graphs of functions:

Proposition 79. *Let X and Y be spaces, and let $f\colon X \to Y$ be continuous. The function $x \mapsto (x, fx)$ defines a homeomorphism between X and the graph*

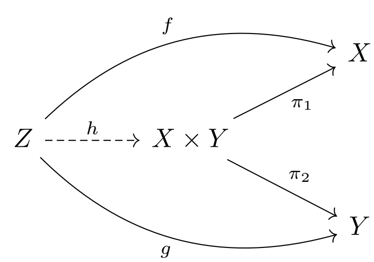
$$\{(x, fx) \mid x \in X\}$$

considered as a subspace of $X \times Y$.

The space $X \times Y$ is a product in the sense of category theory:

Proposition 80. *Suppose $f : Z \to X$ and $g : Z \to Y$ are two continuous functions. Then the function $z \mapsto (fz, gz)$ is a continuous function $Z \to X \times Y$. Moreover, it is the unique function $h\colon Z \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ where π_1 and π_2 are the projection functions.*

We can express this proposition with the following "commutative diagram".



We say that such a diagram "commutes" if given two paths from one space to another, the corresponding compositions of functions are equal. In this case we have two paths from Z to X, one via h followed by π_1 and another via f alone. So we would want $\pi_1 \circ h = f$. Similarly we would want $\pi_2 \circ h = g$ for the diagram to commute. The given continuous functions are indicated by solid arrows, and the continuous function for which we assert existence and uniqueness is indicated with a dotted arrow.

Such commutative diagrams are common in mathematics for a variety of categories. In our case the category is the category of topological spaces where the objects are topological spaces and the "arrows" or "morphisms" are continuous functions. The particular diagram given above is a description of a "product" of objects in a given category. So we say that $X \times Y$ (together with π_1 and π_2) form a product in the category of topological spaces.

14 Disjoint Unions: A Special Case (optional)

There is a duality between open and closed sets: they behave analogously as long as we remember to swap intersections and unions. Similarly there is a duality between the ideas of closures and interiors. A third interesting duality occurs between Cartesian products and disjoint unions.

Suppose X and Y are topological spaces we can form a natural topology on (any model of) the disjoint union $X \sqcup Y$. In order to keep this discussion short and concrete, we stick to the case where the underlying sets of X and Y are given as disjoint sets. In this case the disjoint union is can be thought of as an ordinary union $X \cup Y$.[9]

So let X and Y be topological spaces having underlying sets (also called X and Y) that are disjoint. Then we define a subset W of $X \cup Y$ to be *open* if and only if $W = U \cup V$ where U is an open subset of X and V is an open subset of Y.

Proposition 81. *The open sets described above give a topological structure to the set $X \cup Y$.*

The subspace topologies are as expected:

Proposition 82. *Let X and Y be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. Then the subspace topology on X, considered as a subspace of $X \cup Y$ gives the same topological structure to X as the original topological structure. Similarly, the subspace topology on Y, considered as a subspace of $X \cup Y$ gives the same topological structure to Y as the original topological structure.*

Proposition 83. *Let X and Y be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. Then X and Y are clopen subsets of $X \cup Y$ (in other words, they are both open and closed).*

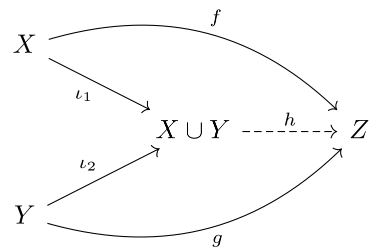
Proposition 84. *Let X and Y be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. If $\mathcal{B}_1$ is a basis for X and $\mathcal{B}_2$ is a basis for Y then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $X \cup Y$.*

Proposition 85. *Let X and Y be topological spaces with disjoint underlying sets, and let $X \cup Y$ be considered as a topological space. Then the inclusion maps $\iota_1 \colon X \to X \cup Y$ and $\iota_2 \colon Y \to X \cup Y$ are continuous open mappings.*

Proposition 86. *Let X and Y be topological spaces with disjoint underlying sets, and let the union $X \cup Y$ be considered as a topological space. Let $\iota_1 \colon X \to X \cup Y$ and $\iota_2 \colon Y \to X \cup Y$ be the inclusion maps. If $f : X \to Z$ and $g : Y \to Z$ are two continuous functions then there is a unique continuous function $h \colon X \cup Y \to Z$ such that $h \circ \iota_1 = f$ and $h \circ \iota_2 = g$.*

[9] In other words, $X \cup Y$ can serve as natural model for $X \sqcup Y$. In general $X \sqcup Y$ is only defined up to homeomorphism, and each actualization is considered to be a "model" for $X \sqcup Y$. If X and Y do not have disjoint underlying sets, then we form a model by replacing X with X' and Y with Y' where X' is homeomorphic to X and Y' is homeomorphic to Y. Then $X' \cup Y'$ with the topology discussed here will form a model for $X \sqcup Y$. Note also that one can generalize this construction from pairs of spaces to infinite families of spaces.

This proposition expresses the claim that the following diagram commutes:



Note the duality between this and the product diagram from the previous section. In category theory this type of diagram is said to characterize a “coproduct”. So we sometimes say that disjoint union is the coproduct in the category of topological spaces.