# Survey of General Topology. Part 1: First Concepts

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This document is the first of a series which surveys the basics of general topology. Other topics such as Hausdorff spaces, metric spaces, connectedness, and compactness will be covered in later documents in the series. The documents in this series are focused on the logical structure of the subject. In an effort to make the logical development more transparent they leave minor details to the reader. I believe that working through these details is an excellent exercise and gives the reader an overall better experience than reading someone else's write-up of such details. Aside from leaving out such minor details, this series is intended to present a full and rigorous developments of the essentials of general topology.

From a logical point of view this series does not require previous experience with topology. It is a self-contained account that can in principle be used as an introduction. So why do I call it a "survey" and not an "introduction"? The term "survey" is used to signal to the reader that this is not intended to replace the many excellent introductory textbooks in topology, and that its structure and purpose are somewhat different. A more introductory account would prepare the student by reviewing the set-theoretical background and other needed background such as key properties of the real numbers. An introductory account would provide more motivation for some of the definitions linking the subject matter to other courses a student is likely to have taken, it would supply the student with multiple exercises to deepen their understanding, and it might explore more counterexamples to explain why hypotheses of theorems are necessary.[1] A more introductory approach might start with metric spaces or subspaces of $\mathbb{R}^n$, and only switch to general spaces when the student was ready to appreciate the additional generality.[2] In short, an introductory text is for student completely new to the subject. This document document, on the other hand, is perhaps more appropriate for someone that has had some exposure to topology in the past and wishes to review the subject from the beginning and deepen their understanding. It also assumes a mathematically mature reader who can supply the simpler proofs or supply minor details to the given proofs (and can accurately assess the correctness of their arguments). It is designed to be more efficient than an introductory textbook, while being at the same time a rigorous account.



[1] Munkres *Topology* [5] is a well-known introductory textbook that has influenced the approach taken in this series.

[2] See for example [2] or [3] for textbooks that start with metric spaces.

So can it be used as a first introduction to general topology? I believe so if used in conjunction with a knowledgeable instructor or knowledgeable friend, or if supplemented with other material.[3]

What is the subject matter? General topology can be viewed as the mathematical study of continuity, connectedness, and related phenomena in a general setting. Such ideas first arise in particular settings such as the real line, Euclidean spaces (including infinite-dimensional spaces), and function spaces. An initial goal of general topology is to find an abstract setting that allows us to formulate results concerning continuity and related concepts (convergence, compactness, connectedness, and so forth) that first arose in these more concrete settings. The basic definitions of general topology are due to Frèchet and Hausdorff in the early 1900's. General topology is sometimes called "point-set topology" since it builds directly on set theory. This is in contrast to algebraic topology, which brings in ideas from abstract algebra to define algebraic invariants of spaces, and differential topology which considers topological spaces with additional structure allowing for the notion of differentiability (general topology only generalizes the notion of continuous functions, not the notion of differentiable function). We approach general topology here as a common foundation for many subjects in mathematics: more advanced topology, of course, including advanced point-set topology, differential topology, and algebraic topology, but also any part of analysis, differential geometry, Lie theory, and just about any subject in modern mathematics from mathematical physics to number theory.

The approach taken here is standard in general topology, but might seem abstract for a beginner. There is a more concrete setting for much of topology built around the concept of a *metric space*, and one can go far in topology using metric spaces only. The more abstract approach to general topology exists partially because one finds that in some situations one needs topological notions outside of the context of a metric space; also the more abstract approach helps clarify the differences between metric properties and general topological properties. For these reasons the general approach is taken here, with metric spaces viewed as particular, but important, examples of a topological space. But a case can be made that beginners should get comfortable with topological ideas in the context of Euclidean spaces, or more generally metric spaces before pursuing the general approach. In summary, this series can be viewed as a concise but rigorous survey of the basics of general topology for the reader that is interested in, and ready for, the general approach.

## 1 Required Background

General topology is a remarkably self-contained subject and does not require much background besides a willingness and ability to think abstractly using set-theoretical reasoning. But it does require a lot of elementary set theory. The reader is expected to be versed in basic logical and set-theoretic techniques employed in the upper-

[3] I did successfully use these notes as a central part of a master's level course in general topology in the Spring of 2021.

division curriculum of a standard mathematics major. But other than that, the subject is self-contained.[4]

I have attempted to give full and clear statements of the definitions and results, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. When I supply a proof it is sometimes just a sketch, but I have attempted to be detailed enough that the prepared reader can supply the details without too much trouble. Even when a rather detailed proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant. There are several examples included and most of these require the reader to work out various details, so they provide additional exercise. In short, I assume the reader is comfortable with mathematical proof utilizing set-theoretical concepts and enjoys the challenge of working out proofs for themselves.

The set-theoretical background assumed here includes the notion of an ordered set which plays an important role in this document. In some examples I assume the notion of a well-ordered set, but more advanced set-theoretic notions such as transfinite ordinal and cardinal numbers are not required.[5] I do assume the reader is familiar with the field of real numbers $\mathbb{R}$, and in some of the examples I assume a little bit of Euclidean geometry in $\mathbb{R}^2$ including a little bit of trigonometry; but trigonometry is not required for the logical development of the subject, only for developing a few (optional) examples.

## 2 Topological Structures. Open and Closed Sets.

Topological spaces are a type of mathematical structure. Other mathematical structures include groups, rings, ordered sets, graphs, and so on. Typically structures consist of a set (sometimes multiple sets) equipped with relations or operators for use in the given set (or sets). In topology the set is equipped with a collection of special subsets called *open sets*. From these we define other aspects of the structure such as the collection of closed sets. There is some degree of arbitrariness in the definition. For instance, we could reverse the definition and start with closed subsets, and define open subsets in terms of closed subsets.

**Definition 1** (Topological Space)**.** Let $X$ be a set, which we call the *underlying set*. A *topological structure* on $X$ is a collection $\mathcal{U}$ of subsets of $X$ called *open subsets* such that the following three "axioms" hold (i) the empty set $\varnothing$ and $X$ as a whole

[4]Set-theoretic reason here is taken to include not just ideas related to intersections, unions, complements, and the empty set, but also functions between arbitrary sets, images and preimages of functions, finite and infinite Cartesian products, relations such as order relations and equivalence relations, well-ordering and so on.

[5]I will comment occasionally on when the axiom of choice is used in a proof. Note that mathematicians usually use this axiom without comment, and without necessarily even being conscious of its use; this practice is fine for most purposes, but some readers might be interested to see how this axiom is used in basic topology.

are open subsets, (ii) for any collection $\mathcal{C} \subseteq \mathcal{U}$ of open subsets, the union $\bigcup_{U \in \mathcal{C}} U$ is an open set, and (iii) the intersection of two open sets is an open set.

A *topological space* is a choice of underlying set $X$ together with a choice of topological structure $\mathcal{U}$ on $X$.

*Remark.* The terminology used in practice is fairly fluid. If $X$ is a set then it is common to use the phrase *a topology on* $X$ to mean a topological structure on the set $X$. The phrase *topological structure* (as opposed to a topological structure on a given set $X$) can also be used to refer to a topological space as a whole.

It is common practice to put the components of a mathematical structure into an ordered tuple. So a topological space can be presented as an ordered pair $(X, \mathcal{U})$ where $X$ is a set and $\mathcal{U}$ is a topological structure on $X$.[6] It is even more common in practice to use the same name for the structure as a whole and the underlying set of the structure. So in group theory the letter $G$ often refers simultaneously to the group as a whole and to the underlying set, where context is used to sort out what is actually meant. The same practice is used in topology where we allow ourselves to use the same name, $X$ say, to refer to the underlying set as well as to the topological space as a whole. Of course this is most appropriate when there is only one topological structure on the set $X$ that is being used in a given context. When we are considering two different topological structures on the same underlying set, our notation should distinguish the two topological spaces from each other. For example, we might specify topological spaces $X_1 = (X, \mathcal{U}_1)$ and $X_2 = (X, \mathcal{U}_2)$ with the same underlying set $X$.

*Remark.* Topology is viewed as a sort of generalized geometry, and so geometric terminology is often used. For example, in this document the term *space* will refer to a topological space. The term *point* will be used to refer to the elements of the underlying set. Similarly we often use the geometric term *map* for functions between spaces (or technically between the underlying sets of the spaces).

In topology it is helpful to adopt notation that distinguishes subsets of the underlying set from sets of such subsets. To that end, if $X$ is the underlying set of a space then we typically employ the term *collection* for a set of subsets of $X$; in other words a collection in this sense is a subset of the power set of $X$. We will usually use script notation for such collections (such as $\mathcal{C}$ or $\mathcal{U}$).

*Remark.* The requirement that the empty set be open is redundant since technically the empty set is the union $\bigcup_{U \in \mathcal{C}} U$ of the empty collection $\mathcal{C} = \varnothing$. However, we are following the lead of many authors who follow the friendlier path of explicitly assuming that the empty set is open.

*Remark.* The axioms of the above definition can be justified using the intuitive notion of "neighborhood". An intuitive notion of "space" usually involves a notion of "neighborhood" where a neighborhood of a point $x$ consists of, at a minimum, the point $x$ together with all points "sufficiently close" to $x$. If the space has a distance, we would want neighborhoods of $x$ to include all points closer than distance $\varepsilon$ from $x$ for some $\varepsilon > 0$. This concept of neighborhood departs a bit from English usage of the term "neighborhood" since this notion of neighborhood is closed under supersets: if $U$ is a neighborhood of $x$ and if $U \subseteq V$ then $V$ is a

[6] Observe that this is technically redundant since the set $X$ can be recovered from $\mathcal{U}$: it is the unique element of $\mathcal{U}$ containing all the elements of $\mathcal{U}$ as subsets.

neighborhood of $x$. (In everyday English you would not say that your whole country is a neighborhood of your house, but the boundary line of how large a neighborhood can be fuzzy in everyday language, and in mathematics there is no need to restrict the maximum size of a neighborhood.)

With the intuitive idea of "neighborhood" we can explain what an open set is supposed to be. Intuitively, an open subset is a subset $U$ of the space such that for all points $x \in U$ the set $U$ is a neighborhood of $x$. From this point of view it makes sense to consider the whole space as open since the whole space is a neighborhood (in fact is the largest neighborhood) of each point in the space. The empty set is vacuously open since it has no points. Also, the union of open sets should be open according to this notion.

If we take this intuitive view of the notion of an open subset, why would the intersection of two open subsets be an open subset? Suppose that $x \in U \cap V$ where $U$ and $V$ are open. Why is the intersection of two neighborhoods of the point $x$ itself a neighborhood? If your space happens to have a distance function then points of distance less than some $\varepsilon_1 > 0$ from $x$ are in $U$ and points of distance less than some $\varepsilon_2 > 0$ are in $V$. So points of distance less than the minimum of $\varepsilon_1$ and $\varepsilon_2$ would be in $U \cap V$. This intuitive justification of the intersection property mentions the concept of "distance", and distances are usually expressed in terms of real numbers. However, it is important to note that the formal definition (Definition 1) does not utilize any concept of distance, and so is independent of the real numbers. However, we just accept that even if one does not have a traditional distance concept, enough of the concept remains to force the intersection of two neighborhoods to be a neighborhood.

In this document *neighborhood* will not be a defined term, at least not in this document, but just a motivating intuitive idea.[7] However, we will go ahead and officially define *open neighborhood*:

**Definition 2.** Let $X$ be a topological space and let $x \in X$. An *open neighborhood* of $x$ is an open subset of $X$ containing $x$.

Next we consider the idea of an "isolated point" and the related notion of "discrete". Intuitively speaking $x \in X$ is isolated from the other points of a space $X$ if the singleton set $\{x\}$ is a neighborhood of $x$. In other words, no other points besides $x$ are needed to form a neighborhood. Given the intuitive description of "open" in terms of neighborhoods, this means $\{x\}$ is open. This intuition is formalized in the following definition:

[7] In Bourbaki's *General Topology* [1] the term *neighborhood* of $x$ is defined to be any set containing as a subset an open set containing $x$, and this definition is not uncommon in topology today. In fact, this definition may be used in some follow-up documents. Clearly such sets should count as neighborhoods, but it is trickier to justify why these are the only neighborhoods using only an intuitive notion of neighborhood.

Bourbaki describes a formal axiomatization of the notion of *neighborhood* where open sets can then be defined in terms of such neighborhoods. Most of Bourbaki's axioms of neighborhood line up with what we intuitively see as a neighborhood, but the last axiom is subtle: if $V$ is a neighborhood of $x$ then there is a neighborhood $W$ of $x$ such that $V$ is a neighborhood of each $y \in W$. After defining an open set to be a set that is a neighborhood to all its points, one can then verify that a set is a neighborhood of $x \in X$ if and only if it contains an open subset containing $x$.

**Definition 3.** Let $X$ be a topological space and let $x \in X$. If $\{x\}$ is open then $x$ is called an *isolated point.* If every point of a space is isolated then the space is said to be a *discrete* space.

*Example* 1. In a discrete space $X$ every singleton subset of $X$ is open, which forces *every* subset of $X$ to be open (by the union axiom). Conversely, if $X$ is a set and we consider $\mathcal{U}$ to consist of all subset of $X$ (the power set of $X$) then $\mathcal{U}$ gives $X$ a topological structure since it clearly satisfies all three axioms, and this space is a discrete space.

At the opposite extreme, if we only declare $X$ and $\varnothing$ to be open ($\mathcal{U} = \{\varnothing, X\}$) then $X$ is also a topological space.[8] Note that if $X$ has at least two elements, this example together with the discrete topological structure gives a quick example of two distinct topological structures on the same topological space.

These first two examples make sense for any underlying set $X$, and use only set-theoretical ideas. Most commonly used topological spaces use geometric or other notions that go beyond just set theory. But there are a few other examples involving basic set theory alone: finite complement topologies (where a nonempty subset is open if and only its complement is finite), countable complement topologies, and so on. We leave it as an exercise to formally define such topologies involving only the concept of cardinality and verify they satisfy the axioms of a topologic space.

Observe that an infinite union of open sets is open. However, we do not usually expect an infinite intersection of open sets to be open. But a finite intersection of open sets will be open:

**Proposition 1.** *The finite intersection of open sets is open.*

The case of the intersection of one or two open sets is covered by requirement (iii) of Definition 1. The general finite case follows, of course, by induction. The statement is valid even for the intersection of zero sets if we define the intersection of the empty collection to be the whole space $X$. In other words, if $\mathcal{C} = \varnothing$ we could define $\bigcap_{U \in \mathcal{C}} U$ to be $X$ using the idea that the intersection of no sets should be "all elements under consideration".[9] We will have no occasion, however, to consider the intersection of zero sets in this document.

Next we consider a simple, but useful, criterion for openness:

**Proposition 2.** *If $U$ is a subset of a space $X$, then $U$ is open in $X$ if and only if for all $x \in U$ there is an open neighborhood $W$ of $x$ contained in $U$.*

*Proof.* One direction follows from the the definition of open neighborhood: take $W = U$. For the other direction let $\mathcal{C}$ be the collection of open sets that are subsets of $U$, and argue that $\bigcup_{W \in \mathcal{C}} W = U$. □

[8] This trivial topology corresponds to the situation where any two points are "arbitrarily close" to each other.

[9] Usually in set theory $\bigcap_{U \in \varnothing} U$ is undefined or problematic in some way, but we can define it to be $X$ if we work in a fixed space $X$. We have already commented that the requirement in Definition 1 that $\varnothing$ be open is redundant, and now we see that the requirement that $X$ be open is also redundant as long as we replace (*iii*) in Definition 1 with (*iii′*) *the finite intersection of open sets is open (including the intersection of the empty collection).* This is the approach of Bourbaki [1].

We end this section with the concept of a *closed set*, which is dual to the idea of an open set.

**Definition 4.** A subset $S$ of a topological space $X$ is *closed* if its complement $X - S$ is open in $X$.

**Proposition 3.** *The whole space and the empty set are closed. The set of closed subsets is closed under arbitrary intersection. The finite union of closed subsets is closed.*

*Remark.* Earlier we motivated the notion of open set using the intuitive notion of neighborhood. Similarly we give a motivation for the notion of closed set using intuitive notions, yielding a notion of closed set that is hopefully more meaningful than merely being the complement of an open set.

Let $x \in X$ where $X$ is a space, and let $S$ be a subset of $X$. We can think of a point $x$ as "touching" $S$ or as being "arbitrarily close" to $S$ if every neighborhood of $x$ intersects $S$. For example, 0 can be thought of as "touching" the open interval $(0, 1)$ of $\mathbb{R}$. Then intuitively a closed set $S$ is one that contains all points touching $S$. In other words, a closed set $S$ is closed under the operation of adding points touching $S$. So the closed interval $[0, 1]$ of $\mathbb{R}$ would be closed, but not the open interval $(0, 1)$ since it excludes 0 and 1. Observe that $S$ is closed in this intuitive sense if and only if every point $x \in X - S$ has a neighborhood not intersecting $S$. In other words every point outside $S$ is "insolated" from $S$ with a neighborhood. Note that $x$ having a neighborhood not intersecting $S$ for all $x \in X - S$ is equivalent to $X - S$ being open (using the intuitive description of open in terms of neighborhoods). Thus our intuitive notion of $S$ being closed is equivalent to the notion of $X - S$ being open.

## 3 Bases

It is often convenient to build open sets out of certain basic sets of a predetermined form. This leads to the notion of a *basis* (plural: *bases*):

**Definition 5.** A *basis* of a topological space $X$ is a collection $\mathcal{B}$ of open subsets of $X$ such that every open set of $X$ is the union of sets from $\mathcal{B}$.

If $B \in \mathcal{B}$ where $\mathcal{B}$ is a basis of $X$ then we say that $B$ is a *basic open set* of $\mathcal{B}$. If $\mathcal{B}$ is fixed in a certain context then we can simply say that $B$ is a *basic open set*. Furthermore, if $x \in B$ where $B$ is such a basic open set then we say that $B$ is a *basic neighborhood of $x$*.

Note that the set of all open sets is obviously a basis. So every topological space has a basis. Usually, however, it is convenient to work with smaller bases made up of basic open sets that are particularly easy to define or to work with. We will see several examples of this below and in further documents, including the collection of open intervals in $\mathbb{R}$ or the collection of open balls in a metric space.

A basis uniquely characterizes a topological structure in the following sense:

**Proposition 4.** *Suppose $X_1$ and $X_2$ are two topological spaces with the same underlying set $X$. Suppose $\mathcal{B}$ is a basis for both $X_1$ and $X_2$. Then $X_1$ and $X_2$ are the* *same topological structure on $X$. In other words, $X_1$ and $X_2$ have the same open sets.*

The definition of basis can be rephrased as follows:

**Proposition 5.** *Let $X$ be a topological space and let $\mathcal{B}$ be a collection of open subsets of $X$. Then $\mathcal{B}$ is a basis for $X$ if and only if, for all $x \in X$ and all open neighborhoods $U$ of $x$, there is a $B$ in $\mathcal{B}$ such that $x \in B \subseteq U$.*

*Proof.* The direction where we assume $\mathcal{B}$ is a basis is straightforward using properties of unions. For the other direction let $U$ be an open set and let $\mathcal{C}$ be the collection of elements of $\mathcal{B}$ contained in $U$; observe that $\bigcup_{B\in\mathcal{C}} B = U$. □

The above characterization of basis is often a bit easier to work with than the original definition, and so it is worth considering in any proof involving a basis. For example it immediately give one direction of the following generalization of our earlier criterion for openness (Proposition 2):

**Proposition 6.** *Let $\mathcal{B}$ be a basis of a space $X$ and let $U$ be a subset of $X$. Then $U$ is open in $X$ if and only if for all $x \in U$ there is a basic open set $B$ of $\mathcal{B}$ such that $x \in B \subseteq U$.*

*Proof.* One direction follows from the previous proposition and the other direction follows from Proposition 2. □

The idea of a basis can be used to *define* a topological structure on a set. To do so, start by proposing a possible basis and try defining a subset to be open if and only if it is the union of subsets in the proposed basis. In order to be careful about this process, let's use the term *potential basis* for a collection of subsets that we propose to be a basis.[10] Here is what we want out of this collection of subsets:

**Definition 6.** A *potential basis* of a set $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that (1) given $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B$, (2) given $x \in A \cap B$ where $A, B \in \mathcal{B}$, there is a $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

*Remark.* The condition (1) is sometimes phrased as saying that "$\mathcal{B}$ is a cover of $X$".

Observe that we do not require that $\mathcal{B}$ be closed under intersection. However, in many examples that follow, $\mathcal{B}$ will in fact be closed under intersection. In this case the second condition is automatic. In other words, the following give sufficient conditions for $\mathcal{B}$ to be a potential basis: (1) $\mathcal{B}$ is a cover of $X$ and (2$'$) $\mathcal{B}$ is closed under pairwise intersection.

If $X$ is already a topological space then we have the following:

**Lemma 7.** *Every basis for a topological space is a potential basis.*

We want to be able to use a potential basis to define a topological structure on a set $X$. The following proposition allows this.

**Proposition 8.** *Given a potential basis $\mathcal{B}$ of a set $X$ there is a unique topological structure on $X$ with $\mathcal{B}$ as a basis.*

[10] The particular term *potential basis* is not standard, but the idea behind it is.

*Proof.* Uniqueness follows from Proposition 4. Let $\mathcal{B}$ be a potential basis and define a collection $\mathcal{U}$ as follows: $U \in \mathcal{U}$ if and only if for each $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. The collection $\mathcal{U}$ is easily seen to satisfy Definition 1 using the definition of potential basis. □

Given a potential basis $\mathcal{B}$ of a set $X$, the corresponding topological space is said to be the topological space *defined by* $\mathcal{B}$.

*Example* 2. In $\mathbb{R}^2$ we can specify two simple potential bases: (i) The collection of open disks, (ii) the collection of interiors of rectangles. Using a few facts from Euclidean geometry we can show that both of these collections are potential bases for $\mathbb{R}^2$. Moreover, using some facts from Euclidean geometry and the following lemma we can show that both of these potential bases define the same topological structure on $\mathbb{R}^2$.

**Lemma 9.** *Two potential bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a given set $X$ define the same topological space if and only if (i) for each $B_1 \in \mathcal{B}_1$ and $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$, and (ii) vice-versa.*

*Remark* (Optional). We can extend the idea of *the topology defined by a collection of subsets* from potential bases to arbitrary covers. Let $\mathcal{C}$ be a cover of $X$ (i.e., suppose $\mathcal{C}$ is a collection of subsets of $X$ such that every $x \in X$ is an element of some $S \in \mathcal{C}$). Observe that the collection of all finite intersections of one or more members of $\mathcal{C}$ forms a potential basis since it is closed under intersections. We call this the *potential basis generated by* $\mathcal{C}$. The topological structure defined by this potential basis is called the *topology generated by* $\mathcal{C}$.

Let $\mathcal{U}$ be the topological structure generated by such a cover $\mathcal{C}$. Then (1) every set of $\mathcal{C}$ is open, in other words the inclusion $\mathcal{C} \subseteq \mathcal{U}$ holds, and (2) if $\mathcal{U}'$ is a topological structure of $X$ with $\mathcal{C} \subseteq \mathcal{U}'$, then $\mathcal{U} \subseteq \mathcal{U}'$. So we can consider $\mathcal{U}$ as the smallest possible topological structure containing $\mathcal{C}$, and so calling $\mathcal{U}$ the topology "generated by $\mathcal{C}$" is appropriate. We should not call this cover $\mathcal{C}$ a basis in general (Munkres [5] calls such a $\mathcal{C}$ a *subbasis*). For instance, observe that if $\mathcal{C}$ is not a potential basis then $\mathcal{C}$ cannot be a basis of the generated topology. On the other hand, observe that when $\mathcal{C}$ is already a potential basis, then the potential basis associated with $\mathcal{C}$ (via finite intersections of elements in $\mathcal{C}$) defines the same topology as $\mathcal{C}$ itself.

## 4 The Order Topology

Now we consider the topology of $X = \mathbb{R}$, or more generally any set $X$ with a given total order. This will serve as a good example where we can use a basis to define a topology.

We use the term *totally ordered set* (or also *linearly ordered set*) to mean a set $X$ equipped with a total order.[11] Throughout this section let $X$ be a totally ordered set.

[11] There are various equivalent definitions of a totally ordered set. For example, one can require a relation $<$ with the following properties: (1) $<$ is transitive, (2) given $x, y$, at least one of the following three holds: $x < y$, $x = y$, $y < x$, and (3) $<$ is anti-reflexive: $\neg(x < x)$ for all $x \in X$. An equivalent definition requires (1) $<$ is transitive, (2′) given $x, y$, exactly one of of the following three holds: $x < y$, $x = y$, $y < x$.

For convenience we will fix two distinct elements not in $X$, which we will call $\infty$ and $-\infty$. We consider the (disjoint) union $X \cup \{-\infty, \infty\}$ which we call the *extended ordered set* and which we denote by $X^*$. We extend the order relation to $X^*$ in the natural way. Observe that $X^*$ is itself a totally ordered set with smallest element $-\infty$ and largest element $\infty$. We sometimes write $+\infty$ for $\infty$ to contrast it with $-\infty$.

Given $a, b \in X^*$, we define the *open interval* $(a, b)$ in the usual way as the set of $x \in X$ such that $a < x < b$. We allow the empty set to be an open interval; for example, if $a \in X$ then $(a, a)$ is empty. Observe that open intervals satisfy the simple law

$$(a, b) \cap (c, d) = \big(\max(a, c), \min(b, d)\big).$$

In particular, the intersection of two open intervals is an open interval. Also note that $(-\infty, \infty) = X$. Thus the set of open intervals forms a potential basis for $X$.

**Definition 7.** Let $X$ be a totally ordered set. Then the *order topology* on $X$ is the topological structure on $X$ defined by the potential basis of open intervals. When $X$ is regarded as a topological space in this way, we call $X$ an *ordered space.*[12]

*Example* 3. In particular, well-ordered sets are topological spaces using the order topology. Observe that for such well-ordered spaces, a point is isolated if and only if it is the first point or the immediate successor of another point. (A point $x$ is isolated means that $\{x\}$ is open.)

For the record we state the following resulting from the definition:

**Proposition 10.** *If $X$ is an ordered space then the collection of open intervals is a basis for $X$. In particular, every open interval of $X$ is open.*

We define *closed intervals* in the usual way. If $a, b \in X^*$ then

$$[a, b] \stackrel{\text{def}}{=} \{x \in X^* \mid a \leq x \leq b\}.$$

We call $[a, b]$ a *closed interval* of $X^*$. If $[a, b]$ is a subset of $X$ then we call $[a, b]$ a *closed interval of* $X$. When $a \leq b$ this occurs if and only if $a, b \in X$ and in this case

$$[a, b] \ = \ \{x \in X \mid a \leq x \leq b\}.$$

Observe that

$$[a, b] \cap [c, d] = \big[\max(a, c), \min(b, d)\big].$$

We call such a relation $<$ a *strict total order* or a *strict linear order*. One can also describe totally ordered sets using $\leq$ instead of $<$. If $<$ is a total order then we define $x \leq y$ in terms of $<$ as $(x < y) \vee (x = y)$. Then $\leq$ is (i) reflexive, (ii) antisymmetric (in the sense that if $x \leq y$ and $y \leq x$ then $x = y$), (iii) transitive, and finally (iv) the relation $\leq$ has the property that $(x \leq y) \vee (y \leq x)$ for all $x, y \in X$. We call this relation $\leq$ a *reflexive total order* or a *nonstrict total order*. Conversely if $\leq$ satisfies these four properties, then we can define $x < y$ as $(x \leq y) \wedge (x \neq y)$, and can show that $<$ is a strict total order in the above sense. Note that if $\leq$ satisfies the first three of these four properties ((i)-(iii) above) then $\leq$ is called a *partial order*. Note also that reflexivity for $\leq$ follows from the fourth property (iv), so requiring it for a reflexive total order is redundant.

[12]So in this document when we use the term "ordered space" for a space $X$, we assume that $X$ is a *totally* ordered set and that the collection of open intervals forms a basis.

**Lemma 11.** *Suppose $a, b \in X$, and let $[a, b]^c$ be the complement of $[a, b]$ in $X$. Then*

$$[a, b]^c = (-\infty, a) \cup (b, \infty).$$

**Proposition 12.** *Let $X$ be an ordered space. Every closed interval of $X$ is a closed subset of $X$.*

We define *half open intervals* $(a, b]$ and $[c, d)$ in the usual way. For example, if $a, b \in X^*$ then

$$[a, b) \stackrel{\text{def}}{=} \{x \in X^* \mid a \leq x < b\}.$$

We define $(a, b]$ in a similar manner. Any open, closed, or half open interval is called an *interval of $X^*$*. If the interval is a subset of $X$ itself then we call it an *interval of $X$*. For example, if $a < b$ then $(a, b]$ is an interval of $X$ if and only if $b \in X$ (i.e., $b \neq +\infty$).

We can expand the intersection formulas above to cover the case of half open intervals. Also, it is understood that if $a \geq b$ then $(a, b) = (a, b] = [a, b) = \varnothing$, and if $a > b$ then $[a, b] = \varnothing$. When we want to emphasize the ordered set $X$ we use subscripts. For example $[0, \infty)_{\mathbb{Z}}$ is the natural numbers (with first element 0) and $[1, n]_{\mathbb{N}}$ is the set $\{1, 2, \ldots, n\}$. We also allow expressions such as $(-\infty, +\infty]_{\mathbb{Z}}$, which is $\mathbb{Z} \cup \{+\infty\}$, but we will rarely need to employ such notation.

Intervals are closely connected to the concept of convexity:

**Definition 8.** A subset $S$ of an ordered set $X$ is said to be *convex* if, for all $x \in X$ and $a, b \in S$,

$$a \leq x \leq b \implies x \in S.$$

*Remark.* An equivalent characterization of a convex subset $S$ of $X$ is that $S$ is convex if and only if, for all $x \in X$, if $x \notin S$ then $x$ is a lower bound of $S$ or $x$ is a upper bound of $S$.

**Proposition 13.** *Every interval in an ordered set $X$ is convex.*

*Remark.* The converse of the above holds for certain ordered sets such as $X = \mathbb{R}$.

**Proposition 14.** *The intersection of convex sets in an ordered set $X$ is itself convex.*

## 5 Closure of a Subset

**Definition 9.** Let $S$ be a subset of a topological space $X$. The *closure* $\overline{S}$ of $S$ in $X$ is the intersection of all closed subsets of $X$ containing $S$.

The closure of $S$ is the smallest closed subset of $X$ containing $S$:

**Proposition 15.** *Let $S$ be a subset of a space $X$. Then $S \subseteq \overline{S}$ and $\overline{S}$ is a closed subset of $X$. Furthermore, if $Z$ is a closed subset of $X$ with $S \subseteq Z$ then $\overline{S} \subseteq Z$.*

From this proposition we obtain the following two consequences:

**Corollary 16.** *A subset $S$ of $X$ is closed if and only if $S = \overline{S}$.*

**Corollary 17.** *Let $A$ and $B$ be subsets of a space $X$. Then*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

The following gives a more concrete descriptions of the closure.

**Proposition 18.** *Let $S$ be a subset of a space $X$. Then $x \in \overline{S}$ if and only if all open neighborhoods $U$ of $x$ intersect $S$.*

Now we focus on the points that are in the closure $\overline{S}$, especially points that might not be in $S$ itself. There are three related concepts that we will use in this series:[13]

**Definition 10.** Let $S$ be a subset of a space $X$.

- A *contact point* or *closure point* is a point $x \in X$ such that every open neighborhood of $x$ intersects $S$. In other words, $x \in \overline{S}$.
- A *limit point* of $S$ is a point $x \in X$ such that every open neighborhood of $x$ intersects $S$ in a point not equal to $x$.
- An *accumulation point* or *$\omega$-accumulation point* of $S$ is a point $x \in X$ such that every open neighborhood of $x$ intersects $S$ in an infinite number of points.

Observe that every accumulation point is a limit point, and every limit point is a contact point. Also contact points are either points in $S$ or limit points of $S$ (or both).

We can use bases for characterizing these sorts of points:

**Proposition 19.** *Let $X$ be a space with basis $\mathcal{B}$. Let $S$ be a subset of $X$ and let $x \in X$.*

- *The point $x$ is a contact (or closure) point of $S$ if and only if every basic open neighborhood of $x$ intersects $S$.*
- *The point $x$ is a limit point of $S$ if and only if every basic open neighborhood of $x$ contains a point of $S$ not equal to $x$.*
- *The point $x$ is an accumulation point of $S$ if and only if every basic open neighborhood of $x$ contains an infnite number of point of $S$.*

[13] The terminology is somewhat variable from source to source. The somewhat misleading term "limit point" as defined here is fairly standard (Munkres [5], Steen and Seebach [6]), but Davis [2] uses the term "cluster point" instead, and Kelley [4] and Willard [7] use the term "accumulation point". As we will see in a later document, our accumulation point and limit points coincide for Hausdorff spaces and so the distinction is only important when we consider non-Hausdorff spaces. Consequently the distinction is not mentioned or emphasized in many sources. But Steen and Seebach [6] does consider the distinction using the term "$\omega$-accumulation point" for our accumulation points.

Gamelin and Green [3] and Steen and Seebach [6] uses "adherent point" for our contact (closure) points, but I could not find a term for limit points in [3]. Some sources (including [6], [7], and [1]) add another condition to those considered here, using the term *condensation point* for points $x \in X$ that have the property that every neighborhood contains an uncountable number of points of $S$.

For convenience, we restate Proposition 18 as follows:

**Proposition 20.** *Let $S$ be a subset of a space $X$. Then $\overline{S}$ is the set of contact points of $S$. In particular, $S$ is closed if and only if it contains all its contact points.*

This helps describe why closed sets are "closed": they are closed under the operation of adding contact points.

For limit points we have the following:

**Proposition 21.** *Let $S$ be subset of a space $X$, let $\overline{S}$ be the closure of $S$, and let $S'$ be the set of limit points of $S$. Then*

$$\overline{S} = S \cup S'.$$

**Proposition 22.** *A subset of a space $X$ is closed in $X$ if and only if it contains all of its limit points.*

## 6 Interior Points and Boundary Points

Next we consider (i) the notion of *interior* which is a dual to the notion of closure, and (ii) the notion of *boundary*.

**Definition 11.** Let $S$ be a subset of a space $X$. The *interior* of $S$ is the union of all open subsets of $X$ contained in $S$. A point in the interior of $S$ is called an *interior point*.

The interior of $S$ is the largest open subset contained in $S$:

**Proposition 23.** *Let $S$ be a subset of a space $X$, and let $I$ be the interior of $S$. Then $I \subseteq S$ and $I$ is open in $X$. Furthermore, if $W$ is an open subset with $W \subseteq S$ then $W \subseteq I$.*

**Proposition 24.** *Let $S$ be a subset of a space $X$ and let $I$ the interior of $I$. Then $S$ is open if and only if $S = I$.*

**Proposition 25.** *Let $S$ be a subset of a space $X$. Then a point $x \in S$ is an interior point if and only if there is an open neighborhood of $x$ contained in $S$.*

**Definition 12.** Let $S$ be a subset of a space $X$. A *boundary point* of $S$ is a point $x \in X$ where every open neighborhood of $x$ contains points inside of $S$ and points outside of $S$. In other words, a boundary point is a point that is both a contact point of $S$ and of its complement $X - S$. The set of boundary points of $S$ is called the *boundary of $S$*.

**Proposition 26.** *Let $S$ be a subset of a space $X$. The boundary of $S$ is the intersection of $\overline{S}$ and $\overline{X - S}$.*

**Proposition 27.** *A set and its complement have the same boundary.*

**Proposition 28.** *Let $S$ be a subset of a space $X$. The closure $\overline{S}$ is the disjoint union of the interior of $S$ and the boundary of $S$. The space $X$ as a whole is the disjoint union of (i) the interior of $S$, (ii) the boundary of $S$, (iii) the interior of the complement*[14] *of $S$.*

**Corollary 29.** *Let $S$ be a subsubset of a space $X$, and let $I$ be the interior of $S$. Then the boundary of $S$ is the difference $\overline{S} - I$.*

**Corollary 30.** *Let $S$ be a subsubset of a space $X$. Then $S$ is closed if and only if it contains its boundary.*

**Corollary 31.** *Let $Z$ be a closed subset of a space $X$. Then $Z$ is the disjoint union of the interior of $Z$ and the boundary of $Z$.*

**Proposition 32.** *Let $A$ be a subset of a space $X$. Then $A$ is both open and closed ("$A$ is clopen") if and only if it has an empty boundary.*

The concept of boundary point has a generalization that will be useful when we consider connectedness (in a later document):

**Definition 13.** Let $S$ and $T$ be subsets of a space $X$. Then a *linking point $x$* for $S$ and $T$ is a point $x \in X$ which is a contact point for both $S$ and $T$. Thus a boundary point is just a linking point for $S$ and $X - S$.

## 7 The Subspace Topology

Any subset of a topological space is automatically itself a topological space using the subspace topology:

**Definition 14.** Let $X$ be a topological space and let $Y$ be a subset of $X$. Then the *subspace* topological on $Y$ is the topology structure on $Y$ created by declaring a subset $V \subseteq Y$ to be open in $Y$ if and only if there is an open subset $U$ of $X$ such that $V = U \cap Y$. We call $Y$ equipped with this topology a *subspace* of $X$.

To legitimize this definition we need to check the conditions of Definition 1:

**Lemma 33.** *The above definition defines a topology on $Y$.*

Closed subsets of a subspace behave similarly to open subsets:

**Proposition 34.** *Let $Y$ be a subspace of $X$. Let $Z$ be a subset of $Y$. Then $Z$ is closed in $Y$ if and only if there is a closed subset $W$ of $X$ such that $Z = W \cap Y$.*

**Proposition 35.** *Suppose $\mathcal{B}$ is a basis of a space $X$. Let $Y$ be a subspace of $X$. Then the collection of sets of the form $B \cap Y$ with $B \in \mathcal{B}$ forms a basis of $Y$.*

[14] Some authors call this the *exterior* of $S$. Note that this is just $X - \overline{S}$. We will not use the term *exterior* in this sense since it could be confused with the complement of $S$.

When considering subsets $A$ of subspaces $Y$ of a space $X$, we have to be careful about phrases such as "$A$ is open" or "$A$ is closed" since there is ambiguity. We should regard open and closed as *relative* notions distinguishing "$A$ open in $X$" from "$A$ open in $Y$" and "$A$ closed in $X$" from "$A$ closed in $Y$". The next two propositions give situations where these notions align, but first a quick definition:

**Definition 15.** Let $X$ be a topological space. If $Y$ is an open subset of $X$ then $Y$ equipped with the subset topology is called an *open subspace* of $X$. If $Y$ is a closed subset of $X$ then $Y$ equipped with the subset topology is called an *closed subspace* of $X$.

**Proposition 36.** *Suppose $Y$ is an open subspace of $X$. Then a subset of $Y$ is open in $Y$ if and only if it is open in $X$.*

**Proposition 37.** *Suppose $Z$ is a closed subspace of $X$. Then a subset of $Z$ is closed in $Z$ if and only if it is closed in $X$.*

There is a kind of transitive law for subspaces:

**Proposition 38.** *Let $Y$ be a subspace of $X$. Let $Z$ be a subset of $Y$. The subspace topology of $Z$ considered as a subset of $Y$ is the same as the subspace topology of $Z$ considered as a subset of $X$.*

Closures are well behaved with respect to subspaces:

**Proposition 39.** *Let $S$ be a subset of $Y$ where $Y$ is a subspace of $X$. Let $\overline{S}$ be the closure of $S$ in $X$. Then the closure of $S$ in $Y$ is $\overline{S} \cap Y$.*

## 8 The Subspace Topology for Order Topologies

If $S$ is a subset of an ordered set $X$ then $S$ can be made into a topological space using the order relation of $X$ in two ways. (1) $S$ can be given the subspace topology considered as a subspace of $X$. Or (2) $S$ can itself be viewed as an ordered set of its own (using the induced order) and so has its own order topology. Note that these two topologies do not always coincide. For example, consider $\mathbb{R}$ with the order topology and let $S = \mathbb{R} - [0, 1)$. Then $[1, \infty)$ is an open subset of $S$ according to the subspace topology. However, it is not open according to the order topology of $S$.

We do have, however, the following lemma and corollary:

**Lemma 40.** *Let $X$ be an ordered set and $S$ a subset. Let $a, b \in S^*$, let $(a, b)_X$ the corresponding open interval in $X$, and let $(a, b)_S$ be the corresponding open interval in $S$. Then*

$$(a, b)_S = (a, b)_X \cap S.$$

*Thus $(a, b)_S$ is open in both the order topology and in the subspace topology of $S$.*

*Remark.* In the above we agree to choose the same extra elements $-\infty$ and $+\infty$ for a subset $S$ as for the original ordered set $X$. This makes $S^*$ into a subset of $X^*$ with order equal to the restriction of the order of $X^*$ to $S^*$. We adopt this convention throughout this section.

**Corollary 41.** *Let $X$ be an ordered set with subset $S$. Then every open subset of $S$ according to the order topology of $S$ is also open in the subspace topology of $S$.*[15]

There is an important case where the two topologies on $S$ do coincide. This is the case where $S$ is convex.

**Lemma 42.** *Let $S$ be a convex subset of an ordered set $X$. If $(a, b)$ is an open interval of $X$ then $(a, b) \cap S$ is an open interval in $S$ according to the order topology of $S$.*

*Proof.* If $(a, b) \cap S$ is empty then the result follows immediately (the empty set is considered an open interval). So assume that $(a, b) \cap S$ is nonempty.

If $a$ is a strict lower bound of $S$ then let $a' = -\infty$, otherwise let $a' = a$. In the second case $a \in S$ by convexity. If $b$ is a strict upper bound of $S$ then let $b' = +\infty$, otherwise let $b' = b$. In the second case $b \in S$ by convexity. Observe that $a', b' \in S^*$ and that $(a', b') \cap S = (a, b) \cap S$.

Let $(a', b')_S$ be the open interval according to the order relation restricted to $S$. By Lemma 40,

$$(a', b')_S = (a', b') \cap S = (a, b) \cap S.$$

□

**Corollary 43.** *Let $S$ be a convex subset of an ordered set $X$. Then the subspace topology of $S$ coincides with the order topology of $S$.*

*Remark.* There are other examples where the subspace topology aligns with the order topology. For instance $\mathbb{Q}$ considered as an ordered set has the same topology as $\mathbb{Q}$ considered as a subspace of $\mathbb{R}$.

Note that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$, and that the following three topologies on $\mathbb{Z}$ are equivalent: (1) the order topology, (2) the subspace topology, and (3) the discrete topology. This follows from the fact that each singleton subset is open in all three topologies.

## 9 Continuous Functions

In topology we do not usually consider general functions, but restrict our attention to functions that “respect” the topological structure in some way. Such functions include continuous functions, open functions, homeomorphisms, and embeddings. The most fundamental of these types of functions are the continuous functions. In fact a key purpose of general topology is to formulate a notion of continuous functions in a general setting.

**Definition 16.** Let $f : X \to Y$ be a function between topological spaces. The function $f$ is said to be *continuous* if the preimage of every open subset of $Y$ is open in $X$.

*Example* 4. Any function from a discrete space into any given topological space is continuous.

[15] Some authors express this by saying that the subset topology on $S$ is *finer* than the order topology, or that the order topology is *courser* than the subset topology.

**Proposition 44.** *Let $X$ be a topological space. The identity map $X \to X$ is continuous. More generally, if $Z$ is a subspace of $X$ then the inclusion map $z \mapsto z$ is a continuous function $Z \to X$.*

**Proposition 45.** *Let $X$ and $Y$ be topological spaces and let $b \in Y$. The constant map $x \mapsto b$ is continuous.*

**Proposition 46.** *The composition of continuous functions is continuous: if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then $g \circ f\colon X \to Z$ is continuous.*

*Remark.* Recall that union and intersection commute with preimage. Moreover, complement commutes with preimage. These facts can be used to prove the following two alternative characterizations of continuity.

**Proposition 47.** *Let $f : X \to Y$ be a function between topological spaces. Suppose we fix a basis for $Y$. Then the function $f$ is continuous if and only if the preimage of every basic open set of $Y$ is open in $X$.*

**Proposition 48.** *Let $f : X \to Y$ be a function between topological spaces. Then $f$ is continuous if and only if the preimage of every closed subset of $Y$ is closed in $X$.*

Here is yet another characterization of continuity:

**Proposition 49.** *Let $f : X \to Y$ be a function between topological spaces. Then $f$ is continuous if and only if the following condition holds for all points $x \in X$ and subsets $A \subseteq X$: if $x$ is a contact point of $A$ then $f(x)$ is a contact point of $f[A]$. In other words, $x \in \overline{A} \implies f(x) \in \overline{f[A]}$, or equivalently $f\left[\,\overline{A}\,\right] \subseteq \overline{f[A]}$.*

*Proof.* Assume $f$ is continuous. Suppose $x \in X$ is such that $f(x) \notin \overline{f[A]}$. Then there is an open subset $V$ of $Y$ containing $f(x)$ which is disjoint from $f[A]$. In particular, $f^{-1}[V]$ is an open neighborhood of $x$ disjoint from $A$. So $x$ cannot be in the closure of $A$.

Conversely, assume $f$ is not continuous. So there is a closed subset $Z$ of $Y$ such that $A \stackrel{\text{def}}{=} f^{-1}[Z]$ is not closed in $X$. Hence there is a contact point $x \in X$ of $A$ such that $x \notin A$. Thus $f(x) \notin Z$ by definition of $A$. Since $f[A] \subseteq Z$ and since $Z$ is closed, $f(x)$ is not a contact point of $f[A]$. Thus $f$ does not preserve contact points. □

Finally we verify that continuity is well-behaved with restriction of domain or restriction of codomain. We start with restriction of domain:

**Proposition 50.** *Suppose $f\colon X \to Y$ is a continuous function between topological spaces. Let $Z$ be a subspace of $X$. Then the restriction $f|_Z\colon Z \to Y$ is continuous.*

*Proof.* The restriction $f|_Z$ is the composition of the inclusion map $Z \to X$ followed by the given function $f\colon X \to Y$. □

Given a function $f\colon X \to Y$, we can always replace the codomain $Y$ with a subset $Z$ of $Y$ as long as $Z$ contains the image of $f$. This map $f'\colon X \to Z$, which differs from $f$ only in what the codomain is designated to be, is called the *restriction of codomain* of $f$ (in particular, $f(x) = f'(x)$ for all $x \in X$). The following asserts that continuity is well-behaved with respect to this operation.

**Proposition 51.** *Suppose $f : X \to Y$ is a function between topological spaces whose image is contained in a subspace $Z \subseteq Y$. Then $f$ is continuous if and only if the corresponding restriction of codomain $f' : X \to Z$ is continuous.*

## 10 Open Mappings

**Definition 17.** Let $f \colon X \to Y$ be a function between topological spaces $X$ and $Y$. The function is said to be an *open mapping* if $f[U]$ is open in $Y$ for all open subsets $U$ of $X$.

*Example* 5. Every function $X \to Y$ is an open mapping if $Y$ is a discrete space.

**Proposition 52.** *Suppose $X$ is a topological space. Then the identity map $X \to X$ is an open mapping.*

**Proposition 53.** *The composition of open mappings is an open mapping.*

**Proposition 54.** *Let $f : X \to Y$ be a function between topological spaces. Suppose we fix a basis $\mathcal{B}$ for $X$. Then the function $f$ is an open mapping if and only if $f[U]$ is open in $Y$ for all basic open sets $U$ in $\mathcal{B}$.*

**Proposition 55.** *Suppose $f \colon X \to Y$ is an open mapping between topological spaces. Let $U$ be an open subspace of $X$. Then the restriction $f|_U$ is an open mapping.*

**Proposition 56.** *Suppose $f \colon X \to Y$ is an open mapping between topological spaces and that $Z$ is a subspace of $Y$ containing the image of $f$. Then the restriction of codomain map $f \colon X \to Z$ is an open map.*

There is a connection between continuous functions and open mappings. Before we make this connection we need to discuss an ambiguity of notation. If $f \colon X \to Y$ is a bijection and if $V$ is a subset of $Y$ then the notation $f^{-1}[V]$ has two possible interpretations: $f^{-1}[V]$ can refer to the preimage of $V$ under $f$, or it can refer to the the image of $V$ under $f^{-1}$. It turns out that both of these interpretations yield the same set, so we can ignore the ambiguity. (If $f$ is not a bijection then there is no ambiguity: the notation refers to the preimage.) We state this as a lemma from set theory:

**Lemma 57.** *Suppose that $f \colon X \to Y$ is a bijection between two sets and that $V$ is a subset of $Y$. Let $A$ be the preimage of $V$ under $f$. Let $B$ be the image of $V$ under $f^{-1} \colon Y \to X$. Then $A = B$. Thus the notation "$f^{-1}[V]$" can be interpreted as referring to both $A$ and $B$.*

Now we can give the connection between continuous functions and open mappings in the case of bijective functions:

**Proposition 58.** *Suppose $f : X \to Y$ is a bijection between topological spaces. Then $f \colon X \to Y$ is continuous if and only if $f^{-1} \colon Y \to X$ is an open mapping.*

*Remark.* We could define the notion of "closed mappings" as well, but we will not have need for such functions in this document. But observe that if $f$ is a bijective open mapping then $f$ must map closed subsets to closed subsets. In fact, for bijective maps the notions of open mapping and closed mapping are equivalent.

## 11 Homeomorphisms

In general, two mathematical structures of a certain kind are "isomorphic" if they are "essentially the same" in the sense that the second can be obtained by replacing the basic elements of the first with the basic elements of the second in such a way that the structure that emerges from this replacement agrees with the given second structure. In topology, an isomorphism is called a *homeomorphism*.

In the context of topological spaces we formulate the notion of "replacing basic elements" of one topological space $X$ with the those of another $Y$ by choosing a bijection $f\colon X \to Y$ of the underlying sets. From set theory we know that $f$ induces a bijection from the power set of $X$ to the power set of $Y$ by the rule

$$U \mapsto f[U].$$

Recall that we defined the structure of a topological space by indicating which subsets are open. So for $f$ to be a topological isomorphism (homeomorphism) we will want $f[U]$ to be open in $Y$ if and only if $U$ is open in $X$; that is to say that the map $U \mapsto f[U]$ should be a bijection between the collection $\mathcal{U}_X$ of open subsets of $X$ and the collection $\mathcal{U}_Y$ of open subsets of $Y$. We codify this idea in the official definition:

**Definition 18.** Let $f : X \to Y$ be a function between topological spaces. We say that $f$ is a *homeomorphism* if $f$ is a bijection with the following property: for all subsets $U \subseteq X$, the subset $U$ is open in $X$ if and only if its image $f[U]$ is open in $Y$.

**Proposition 59.** *Let $f : X \to Y$ be a bijection between topological spaces. Then $f$ is a homeomorphism if and only if $f$ is an open mapping that is continuous.*

*Proof.* Clearly a homeomorphism is an open mapping. To establish continuity of a homeomorphism consider the set-theoretic law:

$$V = f[U] \iff U = f^{-1}[V]$$

which applies to all subsets $U \subseteq X$ and $V \subseteq Y$. The above law also helps to establish that an open continuous bijection is a homeomorphism. $\square$

Here is a commonly found description of homeomorphisms (based on Proposition 58).

**Proposition 60.** *Let $f : X \to Y$ be a function between topological spaces. Then $f$ is a homeomorphism if and only if (1) $f$ is bijective, (2) $f$ is continuous, and (3) the inverse $f^{-1}$ is continuous.*

Similarly Proposition 58 yields the following:

**Proposition 61.** *Let $f : X \to Y$ be a function between topological spaces. Then $f$ is a homeomorphism if and only if (1) $f$ is bijective, (2) $f$ is an open mapping, and (3) the inverse $f^{-1}$ is an open mapping.*

Here are some basic results about homeomorphism.

**Proposition 62.** *Suppose $X$ is a topological space. Then the identity $X \to X$ is a homeomorphism.*

**Proposition 63.** *The composition of homeomorphisms is a homeomorphism.*

**Proposition 64.** *The inverse of a homeomorphism is a homeomorphism.*

*Example* 6. Not every continuous bijection is a homeomorphism. We start with a very natural example: there is a bijective continuous map, using the idea of *angle*, from the interval $[0, 1)_{\mathbb{R}}$ to the unit circle $S^1 \subseteq \mathbb{R}^2$. This function turns out not to be a homeomorphism. Developing this example will require us to take as given some properties of the basic trigonometric functions, including their continuity.[16]

- As we will see later in this document, we can describe the topology on the Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ in terms of the standard topology of $\mathbb{R}$ (the ordered topology considered above). The continuity of the trigonometric functions implies that the function $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by

  $$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

  is continuous.

- Properties of trigonometric functions implies that the image of the above map $\gamma$ is the unit circle $S^1$, and that $t_1, t_2 \in \mathbb{R}$ map to the same point of $S^1$ under $\gamma$ if and only if $t_1 - t_2 \in \mathbb{Z}$.

- The above map $\gamma$ maps $0 \in \mathbb{R}$ to $(1, 0) \in \mathbb{R}^2$, and the continuity of $\gamma$ assures us that the preimage of every neighborhood of $(1, 0)$ contains negative elements $t \in \mathbb{R}$ close to 0. In particular, such a preimage contains a $t$ with $-1/2 < t < 0$. By properties of the sine function applied to such $t$, we see that every neighborhood of $(1, 0) \in \mathbb{R}^2$ contains points $(x, y)$ on the circle $S^1$ with $y < 0$.

- Let $\alpha\colon [0, 1) \to S^1$ be the restriction of $\gamma$, restricting both the domain and codomain. Results about restriction of domain and codomain assures us that this map $\alpha$ is continuous. Note that in fact the map $\alpha$ is a continuous bijection. (We use the subspace topology on $[0, 1)_{\mathbb{R}}$ and $S^1$, but by convexity the topology on $[0, 1)$ is just the order topology.)

- The image of the open subset $[0, 1/2)$ of $[0, 1)_{\mathbb{R}}$ contains only points $(x, y) \in S^1$ with $y \geq 0$, but as we observed every neighborhood of $(1, 0) \in S^1$ contains points $(x, y)$ of $S^1$ with $y < 0$. So $\alpha$ is not an open map.

[16] Future results of this series are not dependent on this example, so we do not need to assume properties about basic trigonometry in our development of basic topology.

- Thus $\alpha\colon [0,1) \to S^1$ is a bijective continuous map that is not a homeomorphism.

*Example* 7. Here is another example of a continuous bijection that is not a homeomorphism. It is more self-contained since it avoids trigonometry, but perhaps it seems more contrived than the previous example. Let $X_1$ be $\mathbb{R} - [0,1)$ with the subspace topology, considered as a subspace of $\mathbb{R}$. Let $X_2$ be the same set but with the order topology. Let $X_1 \to X_2$ be the identity map. Then by Corollary 41 this is a continuous bijection. However, it is not a homeomorphism since $[1,\infty)$ is open in $X_1$ but not in $X_2$.

When we use order topologies, order preserving bijective functions provide examples of homeomorphisms.

**Proposition 65.** *Let $f : X \to Y$ be an order-preserving bijective function between ordered spaces. Then $f^{-1}$ is order-preserving. The image of an open interval is an open interval, and the preimage of an open interval is an open interval. In particular, $f$ is a homeomorphism.*

*Example* 8. Modulo some basic facts about linear functions, we we can use this proposition to show that all nonempty open intervals of $\mathbb{R}$ are homeomorphic. In fact, modulo some facts from algebra and calculus, we can show that the function $x \mapsto x/(1-x^2)$ is an order preserving bijection between $(-1,1)$ and $\mathbb{R}$, and so it is a homeomorphism. We can extend this to show that all open intervals of $\mathbb{R}$ are homeomorphic to $\mathbb{R}$ itself.

Here is a tool to show certain spaces are homeomorphic. We will use this later when discussing product topologies. It shows that even when we start with "one-sided" invertibility, continuity of both functions is enough to yield a homeomorphism.

**Proposition 66.** *Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are continuous maps such that $g \circ f$ is the identity map $X \to X$. Let $Y'$ be the image of $f$. Then the restricted of codomain map $f'\colon X \to Y'$ is a homeomorphism with inverse $g\,|_{Y'}$.*

*Proof.* We have established that $f'$ and $g\,|_{Y'}$ are continuous. Using set-theoretical reasoning, we see that $f'$ and $g\,|_{Y'}$ are bijections and are inverses of each other. Thus $f'$ and its inverse $g\,|_{Y'}$ are continuous, and so $f'$ is a homeomorphism. $\square$

Here are some other useful facts.

**Proposition 67.** *Suppose that $f\colon X \to Y$ is a homeomorphism between spaces. Suppose $A$ is a subspace of $X$, and let the image $B = f[A]$ be considered as a subspace of $Y$. Then $f$ restricts to a homeomorphism $A \to B$.*

*In fact, there is a unique map $A \to f[A]$ making the following commute*[17]*, and this map is the restriction described above, and so is a homeomorphism:*

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d] & f[A] \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

*where the vertical maps are the inclusion maps.*

**Proposition 68.** *Suppose that $f\colon X \to Y$ is a bijection where $X$ is a topological space and where $Y$ is just a set initially without a given topological structure. Define a set in $Y$ to be "open" if it is the image of an open set of $X$ under $f$. Then this proposed collection of open sets gives a topological structure to $Y$, and it is the unique topological structure such that $f$ is a homeomorphism.*

## 12 Embeddings (optional)

Another type of map common in topology is the embedding.[18]

**Definition 19.** An *embedding* $f\colon X \to Y$ between topological spaces is a injective continuous function with the additional property that the restriction of codomain map $f'\colon X \to f[X]$ is a homeomorphism. (We can state this additional property as requiring that $f'$ be an open mapping since $f'$ must already be a continuous bijection.)

You can think of embeddings as a natural generalization of inclusion maps. In fact, we have the following two propositions (the first being, in some sense, a restatement of the definition, and the second is being a special case of the first):

**Proposition 69.** *A map $f\colon X \to Y$ between topological spaces is an embedding if and only if it can be factored as a homeomorphism to a subspace $Y'$ of $Y$ followed by the inclusion map of the subspace $Y'$ into $Y$.*

**Proposition 70.** *Suppose $Y$ is a subspace of a topological space $X$. Then the inclusion map $Y \to X$ is an embedding.*

*Example* 9. If $f\colon X \to Y$ is a homeomorphism, then the restriction of $f$ to any subspace $A$ is an embedding $A \to Y$ (by Proposition 67).

On the other hand, the map $f\colon [0,1)_{\mathbb{R}} \to \mathbb{R}^2$ of Example 6 defined by the rule $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is not an embedding: its restriction of codomain is a continuous bijection that is not a homeomorphism. So a continuous injective mapping can fail to be an embedding.

[17]Here "Commuting" just means that this map $A \to f[A]$ followed by composition with the inclusion $f[A] \to Y$ is the same function as the inclusion $A \to X$ followed by composition with the given $f$. See the discussion at the end of Section 15 for further information about commutative diagrams.

[18]Sections, such as this, that are marked optional can be skipped in the first reading. If these sections are required in a later document, I will alert the reader who can then go back and read the previously optional section if necessary.

This class of embeddings is closed under compositions:

**Proposition 71.** *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are embeddings. Then the composition $X \to Z$ is an embedding.*

*Proof.* Let $f'\colon X \to f[X]$ and $g'\colon Y \to g[Y]$ be restrictions of codomain of $f$ and $g$ respectively. By definition of *embedding*, $f'$ and $g'$ are both homeomorphisms. Consider the following commutative diagram[19] where all the vertical maps are inclusion maps:

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, equal] & f[X] \arrow[r, dashed, "g''"] \arrow[d] & g[f[X]] \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] \arrow[d, equal] & g[Y] \arrow[d] \\
 & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Here the map $g''\colon f[X] \to g[f[X]]$ is the homeomorphism describe in Proposition 67. Observe that the composition of the top row of this diagram is the restriction of codomain of $g \circ f$ to its image, and is a homeomorphism. Thus $g \circ f\colon X \to Z$ is an embedding by Proposition 69. □

Continuous injections that are open maps are automatically embeddings:

**Proposition 72.** *If $f\colon X \to Y$ is an injective continuous map that is an open map. Then $f$ is an embedding.*

*Proof.* The restriction of codomain $X \to f[X]$ is a continuous bijection. It is an open map as well (Proposition 56) so is a homeomorphism. □

*Remark.* Finally we mention that Proposition 66 gives a necessary condition for a function $f\colon X \to Y$ to be an embedding.

## 13 Continuity at a Point

There is a notion of continuity at a point that almost as central as the notion of continuity itself. Here is a traditional definition:

**Definition 20.** Suppose that $f\colon X \to Y$ is a function between topological spaces. We say that $f$ *is continuous at the point* $x_0 \in X$ if for all open neighborhoods $V$ of $f(x_0)$ there is an open neighborhood $U$ of $x_0$ such that $f[U] \subseteq V$.

Often the definition is stated in terms of open subsets of a special form, such as open balls in a metric space. To make this precise we need the notion of a neighborhood basis:

[19]If the concept of "commutative diagram" is unfamiliar then see the discussion near the end of Section 15 for a discussion of this concept.

**Definition 21.** Suppose $X$ is a topological space and let $x \in X$ be a point. A *neighborhoood basis for* $x$ is a collection $\mathcal{B}$ of open neighborhoods of $x \in X$ with the following property: if $U$ is an open neighborhood of $x$ then there is an open neighborhood $B$ of $x$ in $\mathcal{B}$ such that $B \subseteq U$.

Using Proposition 5 we get the following:

**Lemma 73.** *Let $X$ be a topological space with basis $\mathcal{B}$. Let $x \in X$, and let*

$$\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$$

*be the associated collection of basic open neighborhood of $x$. Then $\mathcal{B}_x$ is a neighborhood basis for $x$.*

Continuity at a point is often described in terms of neighborhood bases:

**Proposition 74.** *Let $f\colon X \to Y$ be a function between topological spaces and let $x_0 \in X$. Let $\mathcal{B}_{x_0}$ be a neighborhood basis of $x_0$, and let $\mathcal{B}_{f(x_0)}$ be a neighborhood basis of $f(x_0)$. Then $f$ is continuous at $x_0$ if and only if for all neighborhoods $V$ in $\mathcal{B}_{f(x_0)}$ there is a neighborhood $U$ in $\mathcal{B}_{x_0}$ such that $f[U] \subseteq V$.*

*Remark.* For example, if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a function and if $x_0 \in \mathbb{R}^2$ is a given point, then the open disks centered at $x_0$ forms a neighborhood basis of $x_0$. Similarly, the open disks centered at $f(x_0)$ forms a neighborhood basis of $f(x_0)$. (We will see this laid out in detail in our later document on metric spaces). So by the above proposition, $f$ is continuous at $x_0$ if and only if for any $\varepsilon > 0$, with associated open disk $V_\varepsilon$ of radius $\varepsilon$ and center $f(x_0)$, there is a $\delta > 0$, with associated open disk $U_\delta$ and center $x_0$, such that $f[U_\delta] \subseteq V_\epsilon$. A similar characterization applies to functions $\mathbb{R} \to \mathbb{R}$, where open intervals replace open disks. These descriptions corresponds to the tradition definitions of continuity at a point given in introductory analysis courses.

We can give an elegant description of continuity at a point in terms of contact points:

**Proposition 75.** *Suppose that $f\colon X \to Y$ is a function between topological spaces and that $x$ is a point of $X$. Then $f$ is continuous at the point $x$ if and only if for all subsets $A \subseteq X$ such that $x$ is a contact point of $A$ we have that $f(x)$ is a contact point of $f[A]$.*

*Proof.* Suppose $f$ is continuous at $x \in X$. Assume $x$ is a contact point of a subset $A \subseteq X$, but $f(x)$ is not a contact point of $f[A]$. Then $f(x)$ has an open neighborhood $V$ disjoint from $f[A]$. By continuity at $x$ there is an open neighborhood $U$ of $x$ whose image is contained in $V$. Since points of $A$ do not have their image in $V$, no points of $U$ are in $A$, contradicting the assumption that $x$ is a contact point of $A$.

Conversely suppose that $f$ is not continuous at $x \in X$. Then there is a neighborhood $V$ of $f(x)$ such that every neighborhood of $x$ has points whose image is not in $V$. So $x$ is a contact point of the set of points $A$ not mapping to $V$ under $f$. However, $f(x)$ is certainly not a contact point of $f[A]$ since $V$ is disjoint from $f[A]$. $\square$

**Proposition 76.** *Suppose that $f: X \to Y$ is a function from a space $X$ to a space $Y$. Then $f$ is continuous if and only if it is continuous at $x$ for all $x \in X$.*

*Proof.* If $f$ is continuous, continuity for each $x \in X$ is straightforward.

Now suppose that $f$ is continuous at each $x \in X$. Let $V$ be an open subset of $Y$. We will use Proposition 2 to show that $f^{-1}[V]$ is open in $X$. Let $x \in f^{-1}[V]$, so $V$ is a neighborhood of $f(x)$. By local continuity at $x$, there is a neighborhood $U$ of $x$ such that $f[U] \subseteq V$. This means $U \subseteq f^{-1}[V]$ as desired. This establishes that $f^{-1}[V]$ is open.

A second easy proof combines Propostion 49 and Proposition 75. $\square$

**Proposition 77.** *Let $f: X \to Y$ and $g: Y \to Z$ be maps between topological spaces. Suppose $f$ is continuous at $x_0 \in X$ and $g$ is continuous at $y_0 = f(x_0) \in Y$. Then the composition $g \circ f$ is continuous at $x_0$.*

*Proof.* See Proposition 75. $\square$

Continuity at a point is a local concept in the following sense:

**Proposition 78.** *Let $f: X \to Y$ be a function between topological spaces and let $U$ be an open neighborhood of $x_0 \in X$. Then $f$ is continuous at $x_0$ if and only if the restriction $f|_U: U \to Y$ is continuous at $x_0$.*

*Proof.* In one direction we do not really need $U$ to be open: suppose $f$ is continuous at $x_0$ and let $U$ be any subspace of $X$ containing $x_0$. Then the composition

$$U \xrightarrow{\iota} X \xrightarrow{f} Y$$

is continuous at $x_0$ by the previous proposition, where $\iota$ is the inclusion map. But the composition is just the restriction $f|_U: U \to Y$.

Now suppose $f|_U: U \to Y$ is continuous at $x_0$ where $f: X \to Y$ is a map between topological spaces and $U$ is an open subspace of $X$. We can now appeal to Definition 20 to show $f$ is continuous at $x_0$, and the fact that any open subset of $U$ is open in all of $X$. $\square$

Combining this proposition with Proposition 76 gives the following:

**Corollary 79.** *Suppose $f: X \to Y$ is a function between topological spaces, and suppose $\mathcal{C}$ is a cover of $X$ by open subsets. In other words suppose every member of $\mathcal{C}$ is an open subset of $X$ and the union over all members of $\mathcal{C}$ gives $X$. If $f|_U: U \to Y$ is continuous for all $U \in \mathcal{C}$ then $f$ is continous.*

## 14 Pasting Together Continuous Functions (optional)

We can use Corollary 79 to paste together continuous functions defined on open subspaces to form a continuous function on the whole space:

**Corollary 80.** *Suppose $X$ and $Y$ are topological spaces, and that $\mathcal{C}$ is a cover of $X$ by open subspaces. For each $U \in \mathcal{C}$ let $f_U: U \to Y$ be a continuous function. If, for each $U, V \in \mathcal{C}$, the functions $f_U$ and $f_V$ agree when restricted to $U \cap V$, then there is a unique continuous function $f: X \to Y$ such that $f|_U = f_U$ for each $U \in \mathcal{C}$.*

*Proof.* The existence and uniqueness of our function $f: X \to Y$ follows from basic set-theoretical principles. We just need to establish the continuity of $f$. But this follows directly from Corollary 79. $\square$

We can extend results like Corollaries 79 and 80 to other types of covers. For example, consider the cover of $\mathbb{R}$ by length one closed intervals $[i, i+1]_{\mathbb{R}}$ where $i$ varies among integers. If you have continuous functions $f_i: [i, i+1]_{\mathbb{R}} \to \mathbb{R}$ for each integer $i$, and if $f_i(i+1) = f_{i+1}(i+1)$ for each integer $i$, then we can paste these functions together to form a continuous function $f: \mathbb{R} \to \mathbb{R}$. We now justify this procedure in some generality.

Further results will be based on the following lemma:

**Lemma 81.** *Let $f: X \to Y$ be a function between topological spaces. Suppose that (1) $Z_1, \ldots, Z_n$ is a finite collection of subspaces of the domain $X$ that cover $X$ in the sense that $X = Z_1 \cup \cdots \cup Z_n$, (2) $x$ is a point of the intersection $Z_1 \cap \cdots \cap Z_n$, and (3) for each $Z_i$ the restriction $f|_{Z_i}: Z_i \to Y$ is continuous at $x$. Then $f: X \to Y$ is continuous at $x$.*

*Proof.* Suppose $x$ is a contact point of a subset $A$ of $X$ (with the intention of using Proposition 75). Let $A_i = A \cap Z_i$. Observe that each point of $A$ is in some $A_i$ by assumption (1). If $x$ is not a contact point of $A_i$ then choose an open neighborhood $U_i$ of $x$ disjoint from $A_i$.

If $x$ is not a contact point of $A_i$ for all $i$, then $U = \bigcap U_i$ is an open neighborhood of $x$ not intersecting $A$, a contradiction. Thus we can assume $x$ is a contact point of some $A_{i_0} = A \cap Z_{i_0}$ (relative to the ambient space is $X$, but this implies that $x$ is actually a contact point of $A_{i_0}$ in the subspace $Z_{i_0}$ using assumption (2) as well as the definition of the subset topology). By assumption (3), $f|_{Z_{i_0}}: Z_{i_0} \to Y$ is continuous at $x$ which means that $f(x)$ is a contact point of $f[A_{i_0}]$ (Proposition 75).

In particular, $f(x)$ is a contact point of the superset $f[A]$. Now use Proposition 75 again to conclude that $f$ itself is continuous at $x$. $\square$

We apply this lemma to prove the following generalization of Corollary 79:

**Proposition 82.** *Let $f: X \to Y$ be a function between topological spaces. Suppose $\mathcal{C}$ is a cover of $X$ with the following property: for each $x \in X$ there is a finite subcollection of $\mathcal{C}$ whose members all contain $x$ and whose union contain an open neighborhood of $x$. If $f|_Z: Z \to Y$ is continuous for all $Z \in \mathcal{C}$ then $f: X \to Y$ is continuous.*

*Proof.* Let $x \in X$. By assumption there is finite set $Z_1, \ldots, Z_k$ of elements of $\mathcal{C}$ containing $x$ and an open neighborhood $U$ of $x$ such that

$$U \subseteq Z_1 \cup \cdots \cup Z_k.$$

For each $i$ let $Z_i' = Z_i \cap U$. By our continuity assumption and an earlier result about restriction, $f|_{Z_i'}: Z_i' \to Y$ is continuous on the subspace $Z_i'$ for all $i$. Also $Z_1', \ldots, Z_k'$ cover the subspace $U$, and $x$ is in each $Z_i'$. So by the previous lemma $f|_U: U \to Y$ is continuous at $x$. By Proposition 78, the original function $f: X \to Y$ is continuous at $x$. This holds for all $x \in X$, so $f: X \to Y$ is continuous by Proposition 76. $\square$

This leads to the following corollary allowing for pasting (with a proof similar to that of Corollary 80):

**Corollary 83.** *Suppose $X$ and $Y$ are topological spaces, and that $\mathcal{C}$ is a cover of $X$ with the following property: for each $x \in X$ there is a finite subcollection of $\mathcal{C}$ whose members all contain $x$ and whose union contain an open neighborhood of $x$.*

*For each $Z \in \mathcal{C}$ let $f_Z \colon Z \to Y$ be a continuous function. If, for each $Z, W \in \mathcal{C}$, the functions $f_Z$ and $f_W$ agree when restricted to $Z \cap W$, then there is a unique continuous function $f \colon X \to Y$ such that $f|_Z = f_Z$ for each $Z \in \mathcal{C}$.*

An important special case of Proposition 82 concerns locally finite covers by closed subspaces (which includes the example mentioned above using closed length one intervals).

**Definition 22.** A cover $\mathcal{C}$ of a space $X$ is called *locally finite* if each $x \in X$ has an open neighborhood that intersects only finitely many members of the cover $\mathcal{C}$.

When a locally finite cover consists of *closed* subspaces, we can choose neighborhoods of each point with the extra property needed to apply Proposition 82:

**Lemma 84.** *Suppose $\mathcal{C}$ is a locally finite cover of the space $X$ by closed subspaces. Then every point $x \in X$ has a neighborhood $U$ that intersects only finitely many members of the cover $\mathcal{C}$, and such that every $Z \in \mathcal{C}$ intersecting $U$ contains the point $x$.*

*Proof.* Fix $x \in X$. Since $\mathcal{C}$ is locally finite there is an open neighborhood $V$ of $x$ that intersects only finitely many members of $\mathcal{C}$. Let $Z_1, \ldots, Z_k$ be the members of $\mathcal{C}$ containing $x$, and let $W_1, \ldots, W_l$ be the members of $\mathcal{C}$ that intersect $V$ but do not contain $x$ (so $k \geq 1$ and $l \geq 0$). Let $U$ be the difference $V - (W_1 \cup \cdots \cup W_l)$ (and so $U = V$ in the case where $l = 0$). Observe that $U$ is an open neighborhood of $x$ with the desired properties. □

From this lemma, Proposition 82, and Corollary 83 we get the following two corollaries:

**Corollary 85.** *Let $f \colon X \to Y$ be a function between topological spaces, and suppose $\mathcal{C}$ is a locally finite cover of $X$ by closed subspaces. If $f|_Z \colon Z \to Y$ is continuous for all $Z \in \mathcal{C}$ then $f$ is continuous.*

**Corollary 86.** *Suppose $X$ and $Y$ are topological spaces, and that $\mathcal{C}$ is a locally finite cover of $X$ by closed subspaces of $X$. For each $Z \in \mathcal{C}$ let $f_Z \colon Z \to Y$ be a continuous function. If, for each $Z, W \in \mathcal{C}$, the functions $f_Z$ and $f_W$ agree when restricted to $Z \cap W$, then there is a unique continuous function $f \colon X \to Y$ such that $f|_Z = f_Z$ for each $Z \in \mathcal{C}$.*

It is worth emphasizing that the above includes finite covers by closed sets, and so we restate the results for this case:

**Corollary 87.** *Let $f \colon X \to Y$ be a function between topological spaces, and suppose $\mathcal{C}$ is a finite cover of $X$ by closed subsets. If $f|_Z \colon Z \to Y$ is continuous for all $Z \in \mathcal{C}$ then $f$ is continuous.*

**Corollary 88.** *Suppose $X$ and $Y$ are topological spaces, and that $\mathcal{C}$ is a finite cover of $X$ by closed subspaces of $X$. For each $Z \in \mathcal{C}$ let $f_Z \colon Z \to Y$ be a continuous function. If, for each $Z, W \in \mathcal{C}$, the functions $f_Z$ and $f_W$ agree when restricted to $Z \cap W$, then there is a unique continuous function $f \colon X \to Y$ such that $f|_Z = f_Z$ for each $Z \in \mathcal{C}$.*

## 15 The Product Topology

Given $X$ and $Y$ two topological spaces, we can give a natural topological structure to the Cartesian product $X \times Y$. In a future document we will also describe the topology of arbitrary, even infinite, Cartesian products, but for now we will restrict to this simpler situation of the product of two spaces $X$ and $Y$.

In this section, we will make extensive use of the following basic set-theoretical identity for $A$ and $Z$ subsets of a set $X$, and $B$ and $W$ subsets of a set $Y$:

$$(A \cap Z) \times (B \cap W) = (A \times B) \cap (Z \times W).$$

In particular, this identity gives a quick proof of the following:

**Lemma 89.** *Let $X \times Y$ be the product of two topological spaces. The collection of subsets of the form $U \times W$, where $U$ is open in $X$ and $W$ is open in $Y$, is closed under finite intersections. In particular, this collection is a potential basis.*

**Definition 23.** The *product topology* on $X \times Y$ is the topology generated by the above potential basis.

We state the following proposition "for the record"; it is an immediate consequence of the definition.

**Proposition 90.** *Let $X$ and $Y$ be spaces. If $U$ is an open subset of $X$ and if $W$ is an open subset of $Y$ then $U \times W$ is an open subset of $X \times Y$.*

There is a corresponding statement for closed subsets:

**Proposition 91.** *Let $X$ and $Y$ be spaces. Suppose $A \subseteq X$ and $B \subseteq Y$. If $A$ and $B$ are closed subsets of their respective spaces then $A \times B$ is closed in $X \times Y$. In general, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Here is a convenient way to get bases for Cartesian products:

**Proposition 92.** *Let $X \times Y$ be the product of two topological spaces. Let $\mathcal{B}_1$ be a basis of $X$ and $\mathcal{B}_2$ be a basis of $Y$. Then the collection of sets of the form $U \times W$ with $U$ in $\mathcal{B}_1$ and $W$ in $\mathcal{B}_2$ is a basis of $X \times Y$.*

*Proof.* This is straightforward using Proposition 5 and the definition of the product topology. $\square$

When one Cartesian product is a subspace of another Cartesian product, the subset topology is well behaved:

**Proposition 93.** *Let $X$ and $Y$ be spaces. Let $Z \subseteq X$ and $W \subseteq Y$ be subspaces. Then the subspace topology on $Z \times W$, considered as a subspace of $X \times Y$, is the same as the product topology on $Z \times W$.*

*Proof.* Let $\mathcal{B}$ be the collection of sets of the form $A \times B$ where $A$ is open in $Z$ and $B$ is open in $W$. Then we can check that $\mathcal{B}$ is a common basis for the two topological structures in question (using Proposition 5 for the subspace topology of $Z \times W$ as a subspace of $X \times Y$). Finally, by Proposition 4 the two topologies agree. $\square$

Now we consider functions related to Cartesian products.

**Proposition 94.** *The function $(x, y) \mapsto (y, x)$ defines a homeomorphism*

$$X \times Y \to Y \times X.$$

*Proof.* Use Proposition 47 (using preimages of basic open subsets) to establish continuity of the map and its inverse. $\square$

The following shows that the projection maps are well-behaved.

**Proposition 95.** *Let $X \times Y$ be a product space of spaces $X$ and $Y$. Then both projection functions $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are continuous and are open mappings.*

A function into a Cartesian product can be resolved into functions for each coordinate, and the continuity of such a function is related to the continuity of the assocated coordinate functions. We begin with a purely set-theoretical result:

**Proposition 96.** *Suppose $f \colon Z \to X \times Y$ is a function into a (binary) Cartesian product. Let $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ where $\pi_1$ and $\pi_2$ are the two projection functions. Then $f$ is given by the rule $z \mapsto (f_1(z), f_2(z))$. So $f$ is determined by $f_1$ and $f_2$.*

*Conversely, given two functions $f_1 \colon Z \to X$ and $f_2 \colon Z \to Y$ with a common domain, the function defined by the rule $z \mapsto (f_1(z), f_2(z))$ is the unique function $f \colon Z \to X \times Y$ such that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.*

The issue of continuity is covered by the following result:

**Proposition 97.** *Let $f \colon Z \to X \times Y$ be a function where $X, Y, Z$ are topological spaces. Let $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ be the corresponding coordinate functions where $\pi_1$ and $\pi_2$ are the two projection functions. Then $f$ is continuous if and only if both $f_1$ and $f_2$ are continuous.*

*Proof.* To show continuity of $f$, first show $f^{-1}[U_1 \times U_2] = f_1^{-1}[U_1] \cap f_2^{-1}[U_2]$. $\square$

From the above propositions we can derive various continuity results. The following is an example:

**Proposition 98.** *If $f \colon X \to X'$ and $g \colon Y \to Y'$ are continuous, then so is the function*

$$X \times Y \to X' \times Y'$$

*defined by $(x, y) \mapsto (fx, gy)$.*

**Proposition 99.** *Let $X$ and $Y$ be spaces. If $y_0 \in Y$ then the function $x \mapsto (x, y_0)$ defines a homeomorphism between $X$ and the subspace $X \times \{y_0\}$ of $X \times Y$. Similarly, if $x_0 \in X$ then the function $y \mapsto (x_0, y)$ defines a homeomorphism between $Y$ and the subspace $\{x_0\} \times Y$ of $X \times Y$.*

*Proof.* See Proposition 66. $\square$

**Proposition 100.** *Let $X$ be a space. The function $x \mapsto (x, x)$ defines a homeomorphism between $X$ and the “diagonal’ $\Delta = \{(x, x) \mid x \in X\}$ considered as a subspace of $X \times X$.*

*Proof.* See Proposition 66. $\square$

The above proposition and its proof generalizes to graphs of functions:

**Proposition 101.** *Let $X$ and $Y$ be spaces, and let $f\colon X \to Y$ be continuous. The function $x \mapsto (x, fx)$ defines a homeomorphism between $X$ and the graph*

$$\{(x, fx) \mid x \in X\}$$

*considered as a subspace of $X \times Y$.*

Results such as Propositions 96 and 97 can be used to show that the space $X \times Y$ is a product in the sense of category theory:

**Proposition 102.** *If $f : Z \to X$ and $g : Z \to Y$ are two continuous functions then the function $z \mapsto (fz, gz)$ is a continuous function $Z \to X \times Y$. Moreover, it is the unique function $h\colon Z \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ where $\pi_1$ and $\pi_2$ are the projection functions.*

We can express this proposition with the following “commutative diagram”.

$$\begin{tikzcd}
 & & X \\
Z \arrow[r, dashed, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & X \times Y \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & Y
\end{tikzcd}$$

We say that such a diagram “commutes” if given two directed paths in the diagram from one space to another, the corresponding compositions of functions are equal. In this case we have two paths from $Z$ to $X$, one via $h$ followed by $\pi_1$ and another via $f$ alone. So we would want $\pi_1 \circ h = f$. Similarly we would want $\pi_2 \circ h = g$ for the diagram to commute. The given continuous functions are indicated by solid arrows, and the continuous function for which we assert existence and uniqueness is indicated with a dotted arrow.

Such commutative diagrams are common in mathematics for a variety of categories. In our case the category is the category of topological spaces where the objects are topological spaces and the “arrows” or “morphisms” are continuous functions. The particular diagram given above is a description of a "product" of objects in a given category. So we say that $X \times Y$ (together with $\pi_1$ and $\pi_2$) form a product in the category of topological spaces.

## 16 Disjoint Unions: A Special Case (optional)

There is a duality between open and closed sets: they behave analogously as long as we remember to swap intersections and unions. Similarly there is a duality between the ideas of closures and interiors. A third interesting duality occurs between Cartesian products and disjoint unions.

Suppose $X$ and $Y$ are topological spaces we can form a natural topology on the disjoint union $X \sqcup Y$. In order to keep this discussion short and concrete, we stick to the case where the underlying sets of $X$ and $Y$ are given as disjoint sets. In this case the disjoint union can be thought of as an ordinary union $X \cup Y$.[20]

So let $X$ and $Y$ be topological spaces having underlying sets (also called $X$ and $Y$) that are disjoint. Then we define a subset $W$ of $X \cup Y$ to be *open* if and only if $W = U \cup V$ where $U$ is an open subset of $X$ and $V$ is an open subset of $Y$.

**Proposition 103.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets. The open sets described above give a topological structure on the union $X \cup Y$. Furthermore, every open subset of $X$ is an open subset of $X \cup Y$, and every open subset of $Y$ is an open subset of $X \cup Y$.*

**Proposition 104.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. Then $X$ and $Y$ are each both open and closed subsets of $X \cup Y$ (in other words, they are each 'clopen' subsets of $X \cup Y$).*

The subspace topologies are as expected:

**Proposition 105.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. Then the subspace topology on $X$, considered as a subspace of $X \cup Y$ gives the same topological structure to $X$ as the original topological structure. Similarly, the subspace topology on $Y$, considered as a subspace of $X \cup Y$ gives the same topological structure to $Y$ as the original topological structure.*

**Proposition 106.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets. Let $X \cup Y$ be considered as a topological space. If $\mathcal{B}_1$ is a basis for $X$ and $\mathcal{B}_2$ is a basis for $Y$ then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $X \cup Y$.*

Now we consider functions related to disjoint unions:

[20] If $X$ and $Y$ intersect we would have to do a bit more work to define the disjoint union; perhaps we can do this in detail in a future document where we focus on the category theoretic aspects of topology. Here is an outline of how that might be done: if $X$ and $Y$ intersect then take a space $Y'$ homeomorphic to $Y$ and disjoint from $X$, perhaps using Proposition 68 to construct such a $Y'$. Then form "a model" of $X \sqcup Y$ by considering $X \cup Y'$ with the topology defined in this section. Different choices of $Y'$ give different models, but the models turn out to be canonically homeomorphic which is good enough most of the time.

**Proposition 107.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets, and let $X \cup Y$ be considered as a topological space. Then the inclusion maps $\iota_1 \colon X \to X \cup Y$ and $\iota_2 \colon Y \to X \cup Y$ are continuous open mappings.*[21]

We end with a universal property whose proof is fairly straightforward:

**Proposition 108.** *Let $X$ and $Y$ be topological spaces with disjoint underlying sets, and let the union $X \cup Y$ be considered as a topological space. Let $\iota_1 \colon X \to X \cup Y$ and $\iota_2 \colon Y \to X \cup Y$ be the inclusion maps. If $f : X \to Z$ and $g : Y \to Z$ are two continuous functions then there is a unique continuous function $h \colon X \cup Y \to Z$ such that $h \circ \iota_1 = f$ and $h \circ \iota_2 = g$.*

This proposition expresses the claim that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "f"] & & \\
 & X \cup Y \arrow[r, dashed, "h"] & Z \\
Y \arrow[ur, "\iota_2"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Note the duality between this and the product diagram from the previous section. In category theory this type of diagram is said to characterize a “coproduct”. So we sometimes say that disjoint union is the coproduct in the category of topological spaces.

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[21] This implies these maps are embeddings (see Proposition 72).

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