Noetherian modules and rings

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This document covers some basic results about Noetherian modules and Noetherian rings. An earlier version of this document was written as a worksheet for my Math 520 course in Fall 2010 where many of the proofs were left as exercises. This version continues the policy of leaving many of the proofs to the reader.

1 Required background

This document is written for readers with some basic familiarity with rings, ideals, and modules. For example, the reader should be familiar with the idea that every ring R is an R-module, and that an ideal in a commutative ring is just a submodule of R.

The reader should be familiar with quotients M/N where M is an R-module with submodule N. Similarly, the reader should be familiar with quotient rings R/I where R is a commutative ring and I is an ideal.

In a few places we refer to Euclidean domains and PIDs. We accept as proved that any Euclidean domain is a PID, and that $\mathbb{Z}$ is a Euclidean domain. We also assume the reader is familiar with polynomials rings $R[X_1, \ldots, X_n]$ with coefficients in a ring R.

2 Definitions and Consequences

Throughout this document, R is assumed to be a commutative ring with unity. Many of the results generalize to noncommutative rings, but the commutative case covers most applications.

Definition 1. An R module M is said to be *Noetherian* if the ascending chain condition (ACC) holds for submodules of M. The ACC is the condition that every ascending chain of submodules eventually stabilizes. In other words, given a countable ascending chain of submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$$

there is an m such that

$$M_m = M_{m+1} = M_{m+2} = \ldots \ .$$

†Version of November 22, 2019 (with minor corrections).

Exercise 1. There is a minor variant of the ascending chain condition that is equivalent to the above version. Since it is equivalent, it can also be referred to as the ACC. Prove that M is Noetherian according to the above definition if and only if the following ACC condition holds: every ascending chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

has only a finite number of proper inclusions. A *proper inclusion* of the chain occurs at index $i > 1$ if $M_i \neq M_{i-1}$.

Exercise 2. There is another popular variant of the ascending chain condition that is equivalent to the above version. Since it is equivalent, it can also be referred to as the ACC. Prove that M is Noetherian according to the above definition if and only if the following version of the ACC holds: there is no infinite properly ascending chain:

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \dots$$

Exercise 3. Prove the following proposition.

Proposition 1. *Suppose M is a Noetherian module, and suppose S is a nonempty collection of submodules of M. Then S contains a maximal element. In other words, there is a submodule $N \in S$ such that $N \subsetneq N'$ fails for all $N' \in S$.*

Exercise 4 (Optional)**.** In the above discussion of the ACC we work with the collection $\mathcal{C}$ of submodules of the module M. We used a partial order on $\mathcal{C}$, namely $\subseteq$.

Formulate the three versions of the ACC in terms of an abstract collection $\mathcal{C}$ with a partial order $\preceq$ (in other words, $\preceq$ is reflexive, transitive, and has the property that $x \preceq y$ and $y \preceq x$ together implies $x = y$ for all $x, y \in \mathcal{C}$). Show that the three versions of ACC are equivalent. Show that they are equivalent to requirement that every nonempty subset S of $\mathcal{C}$ has a maximal element (as in Proposition 1).[1]

Show that if $\mathcal{C}$ and $\preceq$ statisfy the ACC, then any subset of $\mathcal{C}$ must as well (where, naturally, we restrict the partial order $\preceq$ to the subset).

Exercise 5. Show that every submodule of a Noetherian module is also Noetherian.

The definition of Noetherian in terms of the ACC is fairly common, but some prefer to define Noetherian in terms of every submodule being finitely generated. The following shows that both approaches are equivalent.

Theorem 2. *Let M be an R-module. Then M is Noetherian if and only if every submodule of M is finitely generated.*

Exercise 6. Prove the above theorem. Hint: for one direction you will need the fact that the union of any ascending chain of submodules is a submodule.

Now we come to the key concept of *Noetherian ring*, which we define in terms of modules.

[1] This is already abstract enough, but we can go further since we do not really need all of the properties of a partial order.

Definition 2. The ring R is said to be a *Noetherian ring* if $M = R$ is Noetherian as an R-module.

Exercise 7. Show that a ring R is Noetherian if and only if the ACC holds for ideals of R.

Exercise 8. Show that a ring R is Noetherian if and only if every ideal of R is finitely generated.

Exercise 9. Show that every field is a Noetherian ring. Show that $\mathbb{Z}$ is a Noetherian ring. Show that every Euclidean domain is a Noetherian ring.

Exercise 10. Show that if R is Noetherian, then any ring isomorphic to R is Noetherian. Show a similar fact for Noetherian modules.

3 Examples of Noetherian Rings and Modules

Commutative algebra is the area of abstract algebra that studies commutative rings. Actually commutative algebra is largely the study of commutative Noetherian rings since most commutative rings of interest in commutative algebra are Noetherian. For example, all PIDs are Noetherian since every ideal of a PID is finitely generated: in fact generated by one element. However, many more rings are Noetherian. The following famous theorem gives an example of how one can form new Noetherian rings from existing Noetherian rings. We will not give the proof here.[2]

Theorem 3 (Hilbert Basis Theorem)**.** *If R is a Noetherian ring, then $R[X]$ is also Noetherian where $R[X]$ is the ring of polynomials with coefficients in R and with symbolic variable X.*

Corollary 4. *If F is a field, then $F[X_1, \ldots, X_n]$ is Noetherian. In general, if R is Noetherian then so is $R[X_1, \ldots, X_n]$.*

Exercise 11. Prove the above corollary. Hint: use the fact that $R[X, Y]$ is isomorphic to $R'[Y]$ where $R' = R[X]$.

Exercise 12. Show that if R is Noetherian, and if I is an ideal of R then the quotient ring R/I is a Noetherian ring.

Exercise 13. Show that if M is a Noetherian R module, and if N is a submodule of M, then M/N is a Noetherian R-module.

Exercise 14. As we showed before, any submodule of a Noetherian module is a Noetherian R-module. Conclude that any ideal of a Noetherian ring R is a Noetherian R-module.

Exercise 15. Suppose that $f : M_1 \to M_2$ is a homomorphism between R-modules. Suppose that f has finitely generated kernel, and finitely generated image. Show that M_1 is finitely generated.

[2] Perhaps I will add a proof in a future version.

Exercise 16. Consider the natural homomorphism $M_1 \times M_2 \to M_2$. Show that the image is M_2 and that the kernel is isomorphic to M_1. Conclude that if M_1 and M_2 are finitely generated, then so is $M_1 \times M_2$.

Exercise 17. Suppose M_1 and M_2 are Noetherian R-modules. Show that $M_1 \times M_2$ is a Noetherian R-module. Hint: If N is a submodule of $M_1 \times M_2$, then consider the restriction of the homomorphism $M_1 \times M_2 \to M_2$ to the subset N of $M_1 \times M_2$. Show that the image and kernel of this restriction are both finitely generated.

Exercise 18. Suppose R is a Noetherian ring. Show that $M = R^n$ is a Noetherian module for all $n \in \mathbb{N}$. (For the case $n = 0$, define R^0 to be the zero module $\{0\}$).

Exercise 19. Suppose that $f : M_1 \to M_2$ is a homomorphism between R-modules. Suppose that f has Noetherian kernel and Noetherian image. Show that M_1 is Noetherian.

4 A theorem for generated modules

Exercise 20. Show that if M is a Noetherian R-module, and if $f : M \to N$ is an R-module homomorphism, then the image of f is also Noetherian. (Hint: use the first isomorphism theorem, and a previous result.)

Theorem 5. *Suppose R is a Noetherian ring. Then an R-module M is Noetherian if and only if it is finitely generated.*

Proof. One direction follows from Theorem 2. For the other direction, suppose that $M = \langle u_1, \ldots, u_n \rangle$. Then the function $(r_1, \ldots, r_n) \mapsto r_1 u_1 + \ldots + r_n u_n$ is a homomorphism $R^n \to M$. The result follows. □

Exercise 21. Fill in the details of the above proof.

5 Factorization in Noetherian integral domains

Let R be an integral domain. An *irreducible element* of R is an element x that is not zero or a unit but has the property that for all $y, z \in R$, the equation $x = yz$ implies y or z is a unit. Irreducible elements play the role in a general integral domain that prime numbers play in $\mathbb{Z}$. It turns out that when R is Noetherian, we get factorization into irreducible elements (but perhaps not unique factorization).

Theorem 6. *Let R be a Noetherian integral domain. Then every nonzero $x \in R$ can be factored as*

$$x = up_1 \cdots p_k$$

where u is a unit in R and where each p_i is irreducible. (We allow $k = 0$ in case x is itself a unit).

Proof. We will appeal to the ACC condition for ideals.[3]

Suppose $x_1 = x$ does not so factor. Then x_1 cannot be a unit, nor can it be an irreducible element. Thus $x_1 = yz$ for some non-units $y, z \in R$. Either y or z must fail to have the desired factorization. Let x_2 be either y or z so that x_2 fails to have the desired factorization. Continue in this way and produce an infinite chain

$$\langle x_1 \rangle \subsetneq \langle x_2 \rangle \subsetneq \langle x_3 \rangle \subsetneq \cdots$$

Such a chain cannot exist. □

Exercise 22. Verify the proof and fill in any details.

[3] Actually, we only need the ACC for *principal* ideals, so the theorem does generalize a bit.