Localization in Integral Domains

A mathematical essay by Wayne Aitken*

Fall 2019†

If R is a commutative ring with unity, and if S is a subset of R closed under multiplication, we can form a ring $S^{-1}R$ of fractions where the numerators are in R and the denominators are in S. This construction is surprisingly useful. In fact, Atiyah-MacDonald, says the following:[1] *"The formation of rings of fractions and the associated process of localization are perhaps the most important technical tools in commutative algebra. They correspond in the algebro-geometric picture to concentrating attention on an open set or near a point, and the importance of these notions should be self-evident."*

Many of the applications of this technique involve integral domains, and this situation is conceptually simpler. For example, if R is an integral domain then the localization $S^{-1}R$ is a subring of the field of fractions of R, and R is a subring of $S^{-1}R$. In an effort to make this a gentle first introduction to localization, I have limited myself here to this simpler situation. More specifically:

- We will consider the localization $S^{-1}R$ in the case where R is an integral domain, and S is a multiplicative system not containing 0.
- We will consider ideals I of R, and construct their localizations $S^{-1}I$ as ideals of $S^{-1}R$.
- More generally, for the interested reader we will consider R-modules, but only those that are R-submodules of the field of fractions K of R. For such R-modules M we will construct $S^{-1}M$ which will be $S^{-1}R$-submodules of K.

(Localization of more general rings and modules, and their universal properties, will be considered in a separate essay.)

This is intended as an introduction, but nevertheless my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So I have only included proofs for results that do not admit a straightforward proof, and often the proofs that are provided are just hints to allow the reader to work out a full proof. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof.

---

†Version of November 10, 2019.

[1] In their classic *Introduction to Commutative Algebra* (1969), by M. F. Atiyah and I. G. MacDonald. This quote is in the introduction to Chapter 3, a chapter devoted to localization

1 Required background

This document is written for readers with some basic familiarity with rings, integral domains, fields, ideals, and modules. This includes familiarity with quotients R/I of rings by ideals. I assume the reader is familiar with the existence of the field of fraction of an integral domain.

I assume familiarity with principal, prime, and maximal ideals, as well as the notion of the ideal (or submodule) generated by a given set. For example, the reader should be familiar with the fact that every proper ideal is contained in a maximal ideal, or that $R/\mathfrak{m}$ is a field if R is a commutative ring with maximal ideal $\mathfrak{m}$. I also assume the reader is familiar with the addition and multiplication operations for ideals. We will use the notation aR or Ra for the principal ideal generated by a (this is sometimes written $\langle a \rangle$). Similarly, $a_1R + \ldots + a_kR$ is the ideal (or submodule) generated by $a_1, \ldots, a_k$ (sometimes written $\langle a_1, \ldots, a_k \rangle$).

Knowledge of R-modules, as opposed to the simpler case of ideals, is somewhat optional. This is provided here to provide support materials for the study of so-called "fractional ideals" in Dedekind Domains and similar rings. So the reader can skip the parts of the document that deal with modules in a first reading. In any case, the only modules we consider are submodules of the field of fractions. (Recall that R is trivially an R-module, and that a R-submodule of R is just an ideal.)

In a few places we assume that the reader is familiar with Noetherian rings (Corollary 13 and then at the end of Section 9). This material can be skipped in a first reading if the reader is unfamiliar with such rings.

Section 9 concerns integrally closed integral domains and Dedekind domains. This section may seem unmotivated if the reader has not studied these concepts before, but at least everything needed about these concepts is given there. In any case, this section can be skipped in a first reading.

2 The basic constructions

Here we provide the main definitions, and some consequences.

Definition 1. A subset S of an integral domain R is said to be a *multiplicative system* if (1) $1 \in S$, (2) $0 \notin S$, and (3) S is closed under multiplication. (In other words, S is a multiplicative submonoid of $R - \{0\}$).

Remark. We really do not need $1 \in S$; just requiring S be nonempty is enough. But it is more comfortable somehow to have $1 \in S$, and adding 1 to S will not change the resulting ring. It is a harmless requirement.

Definition 2. Let S be a multiplicative system of an integral domain R. Let K be the field of fractions of R. Then $S^{-1}R$ is defined to be the set of all elements of K that can be written in the form r/s with $r \in R$ and $s \in S$. We call $S^{-1}R$ the *localization of R with respect to S*.

Proposition 1. *Let R be an integral domain with field of fractions K, and let S be a multiplicative system of R. Then $S^{-1}R$ is a subring of K. In particular, $S^{-1}R$ is an integral domain with subring R.*

Example 1. If $a \in R$ is nonzero then

$$S = \{a^n \mid n \in \mathbb{N}\}$$

is a multiplicative system.[2] Similarly, if $a_1, \ldots, a_k \in R$ are nonzero, then

$$S = \{a_1^{n_1} \cdots a_k^{n_k} \mid n_i \in \mathbb{N}\}$$

is a multiplicative system.

Example 2. Suppose $\mathfrak{p}$ is a prime ideal of R, Then $S = R \smallsetminus \mathfrak{p}$ is a multiplicative system. In this case $S^{-1}R$ is written $R_\mathfrak{p}$, and is called the localization at $\mathfrak{p}$.

Example 3. Since R is an integral domain, $S = R \smallsetminus \{0\}$ is a multiplicative system. In this case $S^{-1}R$ is the field of fractions.

Definition 3. Let S be a multiplicative system of an integral domain R, and let I be an ideal of R. Then $S^{-1}I$ is defined to be the set of elements of $S^{-1}R$ that can be written in the form a/s with $a \in I$ and $s \in S$.

Proposition 2. *Let R be an integral domain with field of fractions K. Let S be a multiplicative system of R, and let I be an ideal of R. Then $S^{-1}I$ is an ideal of $S^{-1}R$. The ideal $S^{-1}I$ is a proper ideal if and only if I is disjoint from S.*

Proof. Recall that an ideal J of a ring is a proper ideal if and only if $1 \notin J$. □

Here is a simple generalization that is sometimes needed (for example, in the theory of so-called "fractional ideals"):

Definition 4. Let R be an integral domain and let K be the field of fractions of R. Recall that K is an R-module. Let M be an R-submodule of K. If S is a multiplicative system of R, then $S^{-1}M$ is defined to be the set of elements of K that can be written in the form $s^{-1}x$ with $x \in M$ and $s \in S$.

Proposition 3. *Let R be an integral domain and let K be the field of fractions of R. If S is a multiplicative system of R and if M is an R-submodule of K then $S^{-1}M$ is an $S^{-1}R$-submodule of K.*

Remark. Let R be an integral domain with field of fractions K. Let S be a multiplicative system of R, and let I be an ideal of R or more generally an R-submodule of K. Then observe that $S^{-1}I$ is just $I(S^{-1}R)$ where

$$I(S^{-1}R) \stackrel{\text{def}}{=} \{ax \mid a \in I, x \in S^{-1}R\}.$$

In the case where $S = R \smallsetminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of R, it is common to write $IR_\mathfrak{p}$ for $S^{-1}I$.

3 Closure of a multiplicative system (Optional)

It is possible for two multiplicative systems to have the same localization. This happens if and only if the closures of the two systems are equal.

[2] Here $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Definition 5. Let S be a multiplicative system of an integral domain R. Then the closure $\overline{S}$ is defined to be the set of all $d \in R$ such that d divides an element of S.

$$\overline{S} \stackrel{\text{def}}{=} \{d \in R \mid de \in S \text{ for some } e \in R\}.$$

Proposition 4. *Let S be a multiplicative system of an integral domain R.*

- $S \subseteq \overline{S}$.
- *The closure $\overline{S}$ is also a multiplicative system of R.*
- $\overline{S} = \overline{\overline{S}}$.
- $S^{-1}R = \overline{S}^{-1}R$
- *Given an inclusion $S_1 \subseteq S_2$ between multiplicative systems, $\overline{S_1} \subseteq \overline{S_2}$.*

Lemma 5. *Let S be a multiplicative system of an integral domain R. Then,*

$$\overline{S} = \left(S^{-1}R\right)^{\times} \cap R.$$

Here is the main result about uniqueness at the level of the closure:

Proposition 6. *Let S_1 and S_2 be two multiplicative systems. Then $S_1^{-1}R \subseteq S_2^{-1}R$ if and only if $\overline{S_1} \subseteq \overline{S_2}$. Thus $S_1^{-1}R = S_2^{-1}R$ if and only if $\overline{S_1} = \overline{S_2}$.*

Proof. Use above lemma to show $S_1^{-1}R \subseteq S_2^{-1}R$ implies $\overline{S_1} \subseteq \overline{S_2}$. □

4 Operations on ideals

The correspondence $I \mapsto S^{-1}I$ is well-behaved with respect to the operations of ideal addition and multiplication.[3]

Proposition 7. *Let I_1, I_2 be ideals of an integral domain R. Let S be a multiplicative system of R. Then in $S^{-1}R$*

$$S^{-1}(I_1 + I_2) = (S^{-1}I_1) + (S^{-1}I_2)$$

and

$$S^{-1}(I_1 I_2) = (S^{-1}I_1)(S^{-1}I_2).$$

The correspondence is also well-behaved with respect to principle ideals:

Proposition 8. *Let S be a multiplicative system of an integral domain R. If $a \in R$ then in $S^{-1}R$*

$$S^{-1}(aR) = a(S^{-1}R).$$

More generally, if U is a set of elements of R, and if I is the ideal generated by U in R, then $S^{-1}I$ is the ideal generated by U in $S^{-1}R$.

[3]Recall that $I_1 + I_2$ is defined as $\{a_1 + a_2 \mid a_1 \in I_1, a_2 \in I_2\}$. Recall that IJ is defined as the set of finite sums $x_1y_1 + \ldots + x_ky_k$ where $x_i \in I$ and $y_i \in J$. These operations result in ideals. Likewise, the intersection of ideals is an ideal.

The correspondence $I \mapsto S^{-1}I$ is well-behaved with respect to intersections:

Proposition 9. *Let I_1, I_2 be two ideals of an integral domain R. Let S be a multiplicative system of R. Then in $S^{-1}R$*

$$S^{-1}(I_1 \cap I_2) = (S^{-1}I_1) \cap (S^{-1}I_2).$$

Exercise 1. Let K be the field of fractions of an integral domain R. Recall K is an R-module. Let M_1, M_2 be R-submodules of K. Define M_1M_2 so that it is a R-submodule of K. Observe that M_1+M_2, M_1M_2, and $M_1 \cap M_2$ are all R-submodules of K. Now show that the natural generalizations of Proposition 7, Proposition 8, and Proposition 9 are valid for R-submodules of K.

Warning: suppose that $a/s \in S^{-1}I$ where a is in an ideal I of R, and s is a multiplicative system S. It does not follow necessarily that $a \in I$. This is illustrated in the following exercise.

Exercise 2. Let $S = \{2^k \mid k \in \mathbb{N}\} \subseteq \mathbb{Z}$, and let $I = 6\mathbb{Z}$. Show that $3/2 \in S^{-1}I$ even though 3 is not in I.

However, if $\mathfrak{p}$ is a prime ideal disjoint from S we can ignore the above warning. See the following exercise.

Exercise 3. Let S be a multiplicative system of an integral domain R. Let $\mathfrak{p}$ be a prime ideal disjoint from S. Show that if $a/s \in S^{-1}\mathfrak{p}$ with $a \in R$ and $s \in S$ then $a \in \mathfrak{p}$.

5 Correspondence of ideals

Let R be an integral domain, and let S be a multiplicative system. As we have seen, the correspondence

$$I \mapsto S^{-1}I$$

sends ideals of R to ideals of $S^{-1}R$, and moreover it sends ideals disjoint from S to proper ideals of $S^{-1}R$.

There is another correspondence, but in the opposite direction. As we will see in the next proposition, the correspondence

$$J \mapsto J \cap R$$

sends ideals of $S^{-1}R$ to ideals of R.

Proposition 10. *Let R be an integral domain, and let S be a multiplicative system. Let J be an ideal of $S^{-1}R$. Then J is a proper ideal if and only if it is disjoint from S. Also, $J \cap R$ is an ideal of R. Finally, if J is a proper ideal, then $J \cap R$ is a proper ideal of R disjoint from S.*

In summary, there are two correspondences

$$\{\text{Ideals of } R\} \to \left\{\text{Ideals of } S^{-1}R\right\}, \qquad I \mapsto S^{-1}I$$

and

$$\{\text{Ideals of } S^{-1}R\} \to \{\text{Ideals of } R\}, \qquad J \mapsto J \cap R$$

A natural question is what happens when we compose these correspondences. Invertibility manifests in one of the two compositions:

Proposition 11. *Let R be an integral domain, and let S be a multiplicative system. Let J be an ideal of $S^{-1}R$. Then*

$$S^{-1}(J \cap R) = J.$$

Since the composition of $J \mapsto (J \cap R)$ followed by $I \mapsto S^{-1}I$ is the identity on the set of ideals of $S^{-1}R$, we get the following:

Corollary 12. *Let R be an integral domain, and let S be a multiplicative system. The correspondence*

$$I \mapsto S^{-1}I$$

is a surjection from the collection of ideals of R to the collection of ideals of $S^{-1}R$.

The correspondence

$$J \mapsto J \cap R$$

is an injection from the collection of ideals of $S^{-1}R$ to the collection of ideals of R.

Recall that a commutative ring R is *Noetherian* if and only if every ideal is finitely generated.

Corollary 13. *Let R be an integral domain, and let S be a multiplicative system of R. If R is Noetherian, then $S^{-1}R$ is Noetherian.*

Proof. Observe that if I is a finitely generated ideal of the form $Ra_1 + \ldots + Ra_k$ then $S^{-1}I$ is just $(S^{-1}R)a_1 + \ldots + (S^{-1}R)a_k$. Since, by the previous corollary, every ideal of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal of R, the result follows. □

The second composition does not, in general, act as the identity map:

Proposition 14. *Let R be an integral domain, and let S be a multiplicative system of R. Let I be an ideal of R. Then*

$$\left(S^{-1}I\right) \cap R = \{b \in R \mid sb \in I \text{ for some } s \in S\}.$$

As we will see, if we restrict the correspondences to prime ideals, the compositions are invertible.

Exercise 4. Let $R = \mathbb{Z}$ with multiplicative system $S = \{2^k \mid k \in \mathbb{N}\}$. Show that the map $I \mapsto S^{-1}I$ is not injective, even on the collection of ideals disjoint from S, by looking at the image of the ideals $3\mathbb{Z}$ and $6\mathbb{Z}$.

Although the surjective map $I \mapsto S^{-1}I$ is not, in general, injective, we can identify a distinguished element in the preimage of any ideal J of $S^{-1}I$:

Proposition 15. *Let S be a multiplicative system of an integral domain R. Let J be an ideal of $S^{-1}R$, and let $\mathcal{J}$ be the set of ideals I of R such that I maps to J under the map $I \mapsto S^{-1}I$. Then $\mathcal{J}$ has a maximum element, under the inclusion relation. In fact, this maximum element is $J \cap R$.*

Proof. Note that $J \cap R \in \mathcal{J}$ by Proposition 11. If $I \in \mathcal{J}$ then $S^{-1}I = J$, which implies $I \subseteq J$. So $I \subseteq J \cap R$. $\square$

6 Correspondence of prime ideals

Let R be an integral domain, and let S be a multiplicative system of R. Now we restrict the two correspondences of the previous section to prime ideals to produce a natural one-to-one correspondence.

Proposition 16. *Suppose $\mathfrak{p}$ is a prime ideal of R disjoint from S. Then $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}R$.*

Proposition 17. *Suppose $\mathfrak{p}$ is a prime ideal of $S^{-1}R$. Then $\mathfrak{p} \cap R$ is a prime ideal of R disjoint from S.*

We know have correspondences

$$\{\text{Prime ideals of } R \text{ disjoint from } S\} \;\to\; \left\{\text{Prime ideals of } S^{-1}R\right\}, \quad \mathfrak{p} \mapsto S^{-1}\mathfrak{p}$$

and

$$\left\{\text{Prime ideals of } S^{-1}R\right\} \;\to\; \{\text{Prime ideals of } R \text{ disjoint from } S\}, \quad \mathfrak{p} \mapsto \mathfrak{p} \cap R$$

Proposition 18. *Let R be an integral domain, and let S be a multiplicative system of R. The rule $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ defines an inclusion preserving bijection*

$$\{\textit{Prime ideals of } R \textit{ disjoint from } S\} \;\to\; \left\{\textit{Prime ideals of } S^{-1}R\right\}.$$

The inverse sends a prime ideal $\mathfrak{p}$ of $S^{-1}R$ to $\mathfrak{p} \cap R$, and is also inclusion preserving.

Proof. The main point is to show that both compositions of $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ and $\mathfrak{p} \mapsto \mathfrak{p} \cap R$ are the respective identity maps. For one composition use Proposition 11. For the other composition, observe that by Proposition 14 we have, for a prime ideal $\mathfrak{p}$ of R,

$$\left(S^{-1}\mathfrak{p}\right) \cap R = \{b \in R \mid sb \in \mathfrak{p} \text{ for some } s \in S\}.$$

If $\mathfrak{p}$ is disjoint from S, then

$$\{b \in R \mid sb \in \mathfrak{p} \text{ for some } s \in S\} = \mathfrak{p}.$$

$\square$

Proposition 19. *Let R be an integral domain, and let S be a multiplicative system of R. Suppose $\mathfrak{m}$ is a maximal ideal of R that is disjoint from S. Then $S^{-1}\mathfrak{m}$ is a maximal ideal of $S^{-1}R$.*

Exercise 5. Show that, in the above proposition, we can weaken the hypothesis that "$\mathfrak{m}$ is a maximal ideal of R that is disjoint from S" to "$\mathfrak{m}$ is a maximal element among prime ideals disjoint from S".

Proposition 20. *Let R be an integral domain, and let S be a multiplicative system of R. Suppose R has the property that every nonzero prime ideal is maximal, then $S^{-1}R$ also has this property.*

Proposition 21. *Let R be an integral domain with prime ideal $\mathfrak{p}$. Let $S = R - \mathfrak{p}$. Then $S^{-1}\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$ is the unique maximal ideal of $S^{-1}R = R_\mathfrak{p}$. Furthermore the group of units of $R_\mathfrak{p}$ is the set of all elements of $R_\mathfrak{p}$ outside the maximal ideal $\mathfrak{p}R_\mathfrak{p}$.*

7 Correspondence of S-large ideals

Let R be an integral domain and let S be a multiplicative system. The correspondence

$$I \mapsto S^{-1}I$$

is a surjection from the collection of ideals of R to the collection of ideals of $S^{-1}R$, but it is not in general an injection. We had better luck when we restricted to prime ideals not intersecting S. Now we consider another class of ideals which yields an injection.

Definition 6. Let S be a multiplicative system of an integral domain. We say that an ideal I is *S-large* if I is only contained in maximal ideals that are disjoint from S.

Proposition 22. *Let S be a multiplicative system of an integral domain R. The collection of S-large ideals of R forms a monoid under products of ideals. In other words, the product of two S-large ideals is S-large, and the ideal R is S-large.*

Exercise 6. Let S be a multiplicative system of an integral domain R. Show that if $\mathfrak{m}$ is a maximal ideal of R that does not intersect S then $\mathfrak{m}$ is S-large. Conclude that $\mathfrak{m}^k$ is also S-large for any $k \geq 1$.

Exercise 7. Let S be a multiplicative system of an integral domain R. Suppose I_1 and I_2 are ideals such that $I_1 \subseteq I_2$. Show that if I_1 is S-large then so is I_2.

Lemma 23. *Let S be a multiplicative system of an integral domain R, and let I be an S-large ideal. Then*

$$(S^{-1}I) \cap R = I.$$

Proof. Clearly $I \subseteq (S^{-1}I) \cap R$. Let $a \in (S^{-1}I) \cap R$. We can write $a = b/s$ where $b \in I$ and $s \in S$. Let $I_a = \{c \in R \mid ac \in I\}$. This is an ideal of R containing I. Suppose that I_a is a proper ideal, and let $\mathfrak{m}$ be a maximal ideal containing I_a. Since $s \in I_a$, we must have that $\mathfrak{m}$ intersects S. But then this contradicts the assumption that I is S-large since $I \subseteq I_a \subseteq \mathfrak{m}$. So I_a is not a proper ideal, but is all of R. Since $1 \in I_a$, we have $a \in I$. □

Proposition 24. *Let R be an integral domain and let S be a multiplicative system. The correspondence*

$$I \mapsto S^{-1}I$$

is an injective homomorphism from the monoid of S-large ideals to the monoid of ideals of $S^{-1}R$.

8 Quotient rings

If $\mathfrak{p}$ is a prime ideal of an integral domain R, and if S is a multiplicative systems of R, how does $R/\mathfrak{p}$ compare to $(S^{-1}R)/(S^{-1}\mathfrak{p})$?

Proposition 25. *Let R be an integral domain with multiplicative system S. Let $\mathfrak{p}$ be a prime ideal of R that is disjoint from S. Then there is a natural injective ring homomorphism*

$$R/\mathfrak{p} \to (S^{-1}R)/(S^{-1}\mathfrak{p})$$

sending $[a]$ to $[a]$. If $\mathfrak{p} = \mathfrak{m}$ is a maximal ideal, then this map is an isomorphism:

$$R/\mathfrak{m} \cong (S^{-1}R)/(S^{-1}\mathfrak{m}).$$

Proof. Start by showing that $\mathfrak{p}$ is the kernel of the natural composition

$$R \to S^{-1}R \to (S^{-1}R)/(S^{-1}\mathfrak{p}).$$

Also, recall that $R/\mathfrak{m}$ is a field if $\mathfrak{m}$ is a maximal ideal. □

Exercise 8. Let $R, S, \mathfrak{p}$ be as in the above proposition, and identify $A = R/\mathfrak{p}$ with a subring of $(S^{-1}R)/(S^{-1}\mathfrak{p})$. Let T be the image of S in A. Show that T is a multiplicative system of the integral domain A and that $(S^{-1}R)/(S^{-1}\mathfrak{p})$ can be identifies with $T^{-1}A$.

Exercise 9. Suppose $\mathfrak{p}$ is a prime ideal of an integral domain R. Show that the integral domain $A = R/\mathfrak{p}$ can be identified with a subring of $F = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, and that under this identification F is the field of fractions of A.

We can extend the isomorphism $R/\mathfrak{m} \cong (S^{-1}R)/(S^{-1}\mathfrak{m})$ of Proposition 25 from maximal ideals $\mathfrak{m}$ to S-large ideals discussed in the previous section.

Lemma 26. *Let I be an S-large ideal of an integral domain R where S multiplicative system of R. If $s \in S$ then the coset $[s]$ is invertible in R/I.*

Proof. Observe that the ideal $I + sR$ is not contained in any maximal ideal. This means that $I + sR = R$. Write $b + sa = 1$ where $b \in I$ and $a \in R$. So $[a]$ is the inverse of $[s]$ in R/I. □

Proposition 27. *Let I be an S-large ideal of an integral domain R where S multiplicative system of R. Then we have a natural isomorphism*

$$R/I \cong S^{-1}R/S^{-1}I, \qquad [a] \mapsto [a].$$

Proof. Consider the composition of natural homomorphisms:

$$R \to S^{-1}R \to S^{-1}R/S^{-1}I.$$

Observe that the kernel is $(S^{-1}I) \cap R$. By Lemma 23 this is I. So we get a natural injective homomorphism

$$R/I \to S^{-1}R/S^{-1}I.$$

Observe that every element of $S^{-1}R/S^{-1}I$ can be written as $[a][s]^{-1}$ where $a \in R$ and $s \in S$. Obviously $[a]$ is in the image of our map. Also $[s]^{-1}$ is in the image by Lemma 26. So $[a][s]^{-1}$ is in the image. We conclude that the natural injection is also surjective. □

Corollary 28. *Let $\mathfrak{m}$ be a maximal ideal of an integral domain. Then we have a natural isomorphism*

$$R/\mathfrak{m}^k \;\cong\; R_{\mathfrak{m}}/(\mathfrak{m}R_{\mathfrak{m}})^k.$$

9 Integrally closed integral domains

Let R be an integral domain with field of fractions K. We say that R is *integrally closed* if, for every monic $f \in R[X]$, every root of f in K is actually in R.

Lemma 29. *Let R be an integrally closed integral domain. Suppose $f \in K[X]$ is a monic polynomial such that $sf \in R[X]$ where $s \in R$ is nonzero. Then every root of f in K is of the form a/s with $a \in R$.*

Proof. We can assume the degree d of of f is at least 1. Since $sf \in R[X]$, we have also $s^d f \in R[X]$. Observe that there is a monic polynomial $g \in R[X]$ such that

$$g(sX) = s^d f(X).$$

Observe also that if $x \in K$ is a root of f then $g(sx) = 0$. Since R integrally closed, $sx = a$ for some $a \in R$. □

Proposition 30. *Let R be an integral domain, and let S be a multiplicative system of R. If R is integrally closed, then so is $S^{-1}R$.*

Proof. Let $x \in K$ be a root of $f \in S^{-1}R[X]$ where f is monic and where K is the field of fractions of R. Let $s \in S$ be such that $sf \in R[X]$. By the lemma, $x = a/s$ for some $a \in R$. Thus $x \in S^{-1}R$. □

Definition 7. An integral domain R is called a *Dedekind domain* if (1) R is integrally closed, (2) every nonzero prime ideal of R is maximal, and (3) R is Noetherian.

Proposition 31. *Let R be a Dedekind domain, and let S be a multiplicative system of R. Then $S^{-1}R$ is a Dedekind domain.*

10 Localizing away from differences

Now we consider the case of related integral domains R_1 and R_2 with the same field of fractions K. The goal is to identify and investigate common multiplicative systems S such that $S^{-1}R_1 = S^{-1}R_2$. (Such considerations are of interest when studying orders in algebraic number fields.)

Lemma 32. *Suppose R_1, R_2 are two integral domains with the same field of fractions. Suppose S is a multiplicative system for both R_1 and R_2 and that*

$$sR_2 \subseteq R_1$$

where $s \in S$. Then

$$S^{-1}R_2 \subseteq S^{-1}R_1.$$

Corollary 33. *Suppose R_1, R_2 are two integral domains with the same field of fractions. Suppose that S is a multiplicative system for R_1 and that*

$$sR_2 \subseteq R_1 \subseteq R_2$$

for some $s \in S$. Then S is also a multiplicative system for R_2 and

$$S^{-1}R_1 = S^{-1}R_2.$$

Corollary 34. *Suppose R_1, R_2 are two integral domains with the same field of fractions. Suppose S is a multiplicative system for R_1 and*

$$sR_2 \subseteq R_1 \subseteq R_2$$

for some $s \in S$. Then $\mathfrak{p} \mapsto \mathfrak{p} \cap R_1$ defines a inclusion preserving bijection between prime ideals of R_2 not intersecting S and prime ideals of R_1 not intersecting S. If $\mathfrak{m}_1$ is a maximal ideal of R_1 not intersecting S, and if $\mathfrak{m}_2$ is the corresponding prime ideal of R_2, then

$$R_1/\mathfrak{m}_1 \cong R_2/\mathfrak{m}_2$$

under an isomorphism sending $[a]$ to $[a]$.

Proof. Let $\mathcal{P}_0$ be the set of prime ideals of $S^{-1}R_1 = S^{-1}R_2$. Let $\mathcal{P}_i$ be the set of prime ideals of R_i disjoint form S. We use Proposition 18 to form a bijection by composition:

$$\mathcal{P}_2 \to \mathcal{P}_0 \to \mathcal{P}_1.$$

This sends $\mathfrak{p}$ first to $S^{-1}\mathfrak{p}$, which is sent in turn to $(S^{-1}\mathfrak{p}) \cap R_1$. But observe that

$$(S^{-1}\mathfrak{p}) \cap R_1 = \mathfrak{p} \cap R_1.$$

Now suppose $\mathfrak{m}_1$ is a maximal ideal of R_1 disjoint from S and that $\mathfrak{m}_2$ is the corresponding prime ideal of R_2. By Proposition 25, we have ring homomorphisms

$$R_1/\mathfrak{m}_1 \to (S^{-1}R_1)/(S^{-1}\mathfrak{m}_1), \quad [a] \mapsto [a]$$

$$R_2/\mathfrak{m}_2 \to (S^{-1}R_2)/(S^{-1}\mathfrak{m}_2), \quad [a] \mapsto [a]$$

The first is an isomorphism since $\mathfrak{m}_1$ is maximal, and the second is at least injective since $\mathfrak{m}_2$ is a prime ideal. Note that $S^{-1}R_1 = S^{-1}R_2$ and $S^{-1}\mathfrak{m}_1 = S^{-1}\mathfrak{m}_2$. Since $R_1 \subseteq R_2$ and the first map is already surjective, the second must also be surjective. Thus $\mathfrak{m}_2$ is maximal, and we get the desired isomorphism. □