Divisors and Krull Domains

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In my essay "What are discrete valuation rings? What are Dedekind domains?" I give an account of many of the classic results about Dedekind domains including the theorem that an integral domain R is a Dedekind domain if and only if every fractional ideal of R is invertible. Recall that a Dedekind domain R is an integral domain satisfying three properties:

1. R is Noetherian.

2. R is integrally closed.

3. Every nonzero prime ideal of R is maximal.

It turns out that much of the theory of Dedekind domains can be carried out with only the first two properties. The basic idea is to replace fractional ideals by equivalence classes of fractional ideals that are called *divisors*. We get invertibility for such divisors when (1) and (2) hold and, in fact, in the more general setting of "completely integrally closed domains". In addition to invertibility, we are interested in unique factorization. In order to have unique factorization of divisors condition (1) and (2) are also sufficient, but we can generalize to completely integrally closed domains that satisfy a weakened version of (1): the ACC not for all ideals of R, but for the so-called *divisorial ideals* (each equivalence class of ideals has a maximum element called its divisorial ideal). Completely integrally closed domains with this weakened version of ACC are called *Krull domains* which are introduced in this essay.

According to B. L. van der Waerden *Algebra*, the theory of divisors presented here started with a publication by van der Waerden himself in 1929, and then was refined by E. Artin. The refined theory appears in the many editions of van der Waerden's famous *Algebra* textbook.[1]

[1]The first edition of van der Waerden's textbook dates from 1930–1931 and was titled *Moderne Algebra*. The fifth edition was published in 1967. My source is the English translation of the second volume of the fifth edition titled *Algebra II* published by Springer–Verlag in 1991, specifically Section 17.7.

We note that van der Waerden does not develop the theory in general Krull domains, but limits himself to Noetherian integral domains that are integrally closed.

The theory of invertibility of divisors starts with the observation that, in an integral domain R with fraction field K, the inverse of a fractional ideal I, if it exists, is given by

$$I^{-1} = \{x \in K \mid xI \subseteq R\}.$$

This definition of inverse will prove critical even at the level of equivalence classes (i.e., divisors). Although some fractional ideals I fail to have a true inverse in a general integral domain, we will still want to use the above ideal which we will call the *formal inverse* and denote by I^{-1} even if it is not the true inverse. In general II^{-1} will not nessarily be the identity ideal R, but $II^{-1} \subseteq R$ holds generally. If we can set up an equivalence relation such that $[II^{-1}] = [R]$ at the level of equivalence classes then we have achieved invertibility.

What should the equivalence relation be? We would hope that it would be faithful with respect to divisibility of elements of R. So xR should be equivalent to yR if and only if $xR = yR$. Note that distinct principal fractional ideals have distinct inverses. However, for fractional ideals in general it is possible that two distinct fractional ideals I, J have the same formal inverse $I^{-1} = J^{-1}$. So if we want an equivalence relation $\sim$ relation where $[I^{-1}]$ is the inverse of $[I]$, it must have the property that $I^{-1} = J^{-1}$ implies $I \sim J$. Van der Waerden's idea is to build an equivalence relation around this constraint and use the equivalence relation $\sim$ where $I \sim J$ if and only if $I^{-1} = J^{-1}$. This turns out to work well for Noetherian integral domains that are integrally closed, or more generally integral domains that are completely integrally closed (see Section 3 for the definition of *completely integrally closed*). The first part of this document is devoted to developing the theory of divisors considered as equivalence classes under this equivalence.

Then we turn to the topic of Krull domains and unique factorization of divisors. We conclude with some basic properties of Krull domains.

I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. My philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So, whenever a proof is not given or is sketchy, this signals to the reader that they should work out the proof and that the details are reasonably straightforward.

1 General background

This is a sequel to my essay *What are discrete valuation rings? What are Dedekind domains?*, so I assume familiarity with some of the material presented there including material concerning Dedekind domains and fractional ideals. Some material on fractional ideals is reviewed in the next section. Starting in Section 7, I assume familiarity with discrete valuations and discrete valuation rings (DVRs). Starting in Section 8, I assume familiarity with localization in an integral domain. These topics can be found in my earlier essays, and in many other places.

We adopts some notation. Let R be an integral domain. If such an R is a Noetherian ring we call it a *Noetherian domain.* If such a ring R is integrally closed we call it an *integrally closed domain.*

2 Fractional ideals (background)

In this section we review, without proof, some basic properties of fractional ideals. Throughout this section let R be an integral domain and let K be its fraction field. A *fractional ideal* I is a nonzero R-submodule of K such that dI is an ideal for some nonzero $d \in R$. By way of contrast, we sometimes call an ideal I of R an *integral ideal*. We note that every finitely generated nonzero R-submodule of K must be a fractional ideal. The converse holds for Noetherian rings, but not for general integral domains.

Given two fractional ideals I and J, we define the fractional ideal IJ as the collection of finite sums of products xy with $x \in I$ and $y \in J$. The collection of fractional ideals forms a commutative monoid under multiplication with identity element being the identity ideal $I = R$. If I is a fractional ideal of R, then we say that I is *invertible* if there is a fractional ideal J such that $IJ = R$.

If R is a Dedekind domain then every fractional ideal I is invertible with inverse given by

$$I^{-1} = \{x \in K \mid xI \subseteq R\}.$$

In particular, $II^{-1} = R$. In a general integral domain, not every fractional ideal is invertible in the sense that $II^{-1} = R$. Although I^{-1} is not the inverse of I if I is not invertible, we still use the notation I^{-1} and we call it the *formal inverse*. In general we have the following properties for I a fractional ideal:

1. $II^{-1} \subseteq R$.
2. I^{-1} is a fractional ideal.
3. I^{-1} is the maximum fractional ideal J (under inclusion) such that $IJ \subseteq R$. In other words, if $IJ \subseteq R$ then $J \subseteq I^{-1}$.
4. If there is a fractional ideal J with the property that $IJ = R$, then $J = I^{-1}$.

If I, J are fractional ideals with $I \subseteq J$ then

$$J^{-1} \subseteq I^{-1}.$$

We will use formal inverses to define an equivalence relation on the monoid of fractional ideals, then show that this equivalence relation is well-behaved under products. Divisors will be defined as equivalence classes under this equivalence relation.

If I is a fractional ideal of R, it may turn out to be a fractional ideal for other rings R' such that $R \subseteq R' \subseteq K$. We can characterize the maximum such ring:

Definition 1. Let R be an integral domain and let K be its fraction field. If I is a fractional ideal, then

$$\mathcal{R}(I) \stackrel{\text{def}}{=} \{x \in K \mid xI \subseteq I\}.$$

Proposition 1. *Let R be an integral domain and let K be its fraction field. If I is a fractional ideal of R, then the following hold:*

- *$\mathcal{R}(I)$ is a subring of K containg R: so $R \subseteq \mathcal{R}(I) \subseteq K$.*

- *I is a fractional ideal of $\mathcal{R}(I)$, where scalar multiplication is induced by the product of K. In fact, $\mathcal{R}(I)$ is the maximum subring R' of K (under inclusion) such that I is a fractional ideal of R'.*
- *$\mathcal{R}(I)$ is a fractional ideal of R.*

3 Completely integral closed domains

In this section let R be an integral domain and let K be its fraction field. Recall that an element $x \in K$ is said to be *integral over* R if x is a root of a monic polynomial in the polynomial ring $R[X]$. We take it as established background knowledge that this holds if and only if $R[x]$ is a finitely generated fractional ideal of R. We say that R is *integrally closed* if every element of K that is integral over R is actually in R. So we have the following:

Proposition 2. *Let R be an integral domain with field of fractions K. Then R is integrally closed if and only if the following holds for all $x \in K$:*

$$\text{If } R[x] \text{ is a finitely generated fractional ideal then } R[x] = R.$$

If we drop the requirement "finitely generated" we have a stronger condition:

Definition 2. Let R be an integral domain with field of fractions K. Then R is said to be *completely integrally closed* if the following holds for all $x \in K$:

$$\text{If } R[x] \text{ is a fractional ideal then } R[x] = R.$$

An integral domain that is completely integrally closed will be called a *completely integrally closed domain.*

We have the following equivalences:

Proposition 3. *Let R be an integral domain with field of fractions K. Then the following are equivalent:*

1. *R is completely integrally closed. In other words, for all $x \in K$, if $R[x]$ is a fractional ideal then $R[x] = R$.*
2. *$\mathcal{R}(I) = R$. for all fractional ideals I of R.*
3. *For all $x \in K$ and all nonzero $d \in R$, if $dx^k \in R$ for all $k \geq 1$ then $x \in R$.*

Proof. (1) $\implies$ (2). Suppose $x \in \mathcal{R}(I)$ for some fractional ideal I. Then $R[x]$ is an R-submodule of the fractional ideal $\mathcal{R}(I)$. So $R[x]$ is itself a fractional ideal of R. Thus $R[x] = R$. Therefore $x \in R$. So $\mathcal{R}(I) \subseteq R$, hence $\mathcal{R}(I) = R$.

(2) $\implies$ (1). Let $I = R[x]$ be a fractional ideal. Then $R \subseteq R[x] \subseteq \mathcal{R}(I) = R$.

(1) $\implies$ (3). Suppose $x \in K$ and $d \in R$ are such that $dx^k \in R$ for all $k \geq 1$ where d is nonzero. Then $dR[x] \subseteq R$. So $I = R[x]$ is a fractional ideal of R. Thus $R[x] = R$. Therefore, $x \in R$.

(3) $\implies$ (1). Assume that $x \in K$ is such that $R[x]$ is a fractional ideal of R. So $dR[x] \subseteq R$ for some nonzero $d \in R$. In particular, $dx^k \in R$ for all $k \geq 0$. By condition (3), $x \in R$. Thus $R[x] = R$. $\square$

Proposition 4. *Suppose that R is a Noetherian domain. Then R is completely integrally closed if and only if R is integrally closed.*

Suppose R is an integral domain. Then R is completely integrally closed implies that R is integrally closed.

Proof. Every fractional ideal of a Noetherian domain is finitely generated. $\square$

The intersection of completely integrally closed domains is completely integrally closed:

Proposition 5. *Let K be a field, and let $(R_i)_{i\in\mathcal{I}}$ be a nonempty family of subrings of K. If each R_i is completely integrally closed, then so is the intersection R.*

Proof. Let K' be the fraction field of R. Suppose that $d \in R$ is nonzero and that $x \in K'$ is such that $dx^k \in R$ for all $k \geq 1$. In particular, for each R_i in the family, $d \in R_i$, x is in the fraction field of R_i, and $dx^k \in R_i$ for all $k \geq 1$. Since R_i is completely integrally closed, $x \in R_i$. So x is in the intersection R. $\square$

4 Inverse-inclusion, inverse-equality, and divisors

We now consider relations on fractional ideals involving formal inverses I^{-1}. In most of this section R will be a general integral domain with fraction field K.

As we have noted, if I, J are fractional ideals with $J \subseteq I$ then $I^{-1} \subseteq J^{-1}$. However, the converse is not true in general. When the converse does hold, it signifies an important "divisibility-like" relation between I and J:

Definition 3. Let I, J be fractional ideals of an integral domain R. We define $I \leq J$ to mean $I^{-1} \subseteq J^{-1}$. We call this relation *inverse-inclusion.*

Proposition 6. *This inverse-inclusion relation is reflexive and transitive.*

Proposition 7. *Let I, J be fractional ideals of an integral domain R. If $J \subseteq I$ then $I \leq J$.*

Proposition 8. *Let I, J be fractional ideals of an integral domain R and let K be the fraction field of R. The following are equivalent:*

1. *$I \leq J$, in other words $I^{-1} \subseteq J^{-1}$.*
2. *For all $x \in K$, if $xI \subseteq R$ then $xJ \subseteq R$.*
3. *For all $x \in K^\times$, if $I \subseteq xR$ then $J \subseteq xR$.*
4. *$JI^{-1} \subseteq R$.*

Proof. The equivalences $1 \iff 2$ and $2 \iff 3$ are straightforward. For $1 \iff 4$, note that J^{-1} is the maximum fractional ideal J' such that $JJ' \subseteq R$. $\square$

Proposition 9. *Let I_1, I_2, J be fractional ideals of an integral domain R. If $I_1 \leq I_2$ then $I_1 J \leq I_2 J$.*

Proof. Since $(I_1J)(I_1J)^{-1} \subseteq R$ we have $J(I_1J)^{-1} \subseteq I_1^{-1}$. However, by assumption $I_1^{-1} \subseteq I_2^{-1}$. Thus $J(I_1J)^{-1} \subseteq I_2^{-1}$. So $(I_2J)(I_1J)^{-1} \subseteq I_2I_2^{-1} \subseteq R$. By the previous proposition $I_1J \leq I_2J$. □

Proposition 10. *Let I be a fractional ideal of an integral domain R. Then $R \leq I$ if and only if I is an integral ideal.*

Proof. This is straightforward given Proposition 8 part 4. □

Proposition 11. *Let I, J be fractional ideals of an integral domain R. If $I \leq J$ then $J \subseteq (I^{-1})^{-1}$.*

Proof. By Proposition 8 part 4 we have $JI^{-1} \subseteq R$. The conclusion follows. □

Corollary 12. *Let I, J be fractional ideals of an integral domain R. If I is invertible then $I \leq J$ if and only if $J \subseteq I$.*

Proof. One direction was noted above (Proposition 7). So assume $I \leq J$ and use the above proposition. □

Definition 4. Let I, J be fractional ideals of an integral domain R. We define $I \sim J$ to mean $I^{-1} = J^{-1}$. We call this relation *inverse-equality.*

Proposition 13. *Let I, J be fractional ideals of an integral domain R. Then $I \sim J$ if and only if $I \leq J$ and $J \leq I$.*

Proposition 14. *Suppose $I \sim J$ where I and J are fractional ideals of an integral domain R. Then I is an integral ideal if and only if J is an integral ideal.*

Proof. This is straightforward given the previous proposition and Proposition 10. □

Proposition 15. *Let R be an integral domain. The inverse-equality relation is an equivalence relation on the set of fractional ideals of R.*

Proposition 16. *Suppose $I_1 \sim I_2$ and $J_1 \sim J_2$ where I_1, I_2, J_1, J_2 are fractional ideals of an integral domain R. Then $I_1 \leq J_1$ if and only if $I_2 \leq J_2$. In addition, $I_1J_1 \sim I_2J_2$.*

Definition 5. Let I, J be fractional ideals of an integral domain R. Let $[I]$ and $[J]$ be the equivalence classes under inverse-equality. Then $[I] \leq [J]$ is defined to be true if and only if $I \leq J$. Similarly, $[I][J]$ is defined to be $[IJ]$. By the above lemma, these definition are well-defined.

Proposition 17. *Let R be an integral domain. The set of equivalence classes of fractional ideals of R under inverse-equality is a commutative monoid with identity $[R]$. This set is partially ordered under the relation $\leq$.*

Definition 6. If I is a fractional ideal of an integral domain R then its equivalence class $[I]$ under inverse-equality is called the *divisor* of I. The collection of such equivalence classes is called the *divisor monoid* of R.

The criterion of Proposition 8 part 3 tells us that all that matters for determining the relation $\leq$, and hence $\sim$, is the principal fractional ideals that contain the given fractional ideals. Observe that in fact the *intersection* of principal ideals containing the given fractional ideals is enough to determine $\leq$. This leads to the following definition:[2]

Definition 7. Let R be an integral domain. Given a fractional ideal I of R, we define $\overline{I}$ to be the intersection of all principal fractional ideals containing I:

$$\overline{I} \stackrel{\text{def}}{=} \bigcap_{I \subseteq xR} xR$$

Proposition 18. *If I is a fractional ideal of an integral domain, then $\overline{I}$ is a fractional ideal containing I. If I is principal then $\overline{I} = I$. Furthermore, I is an integral ideal if and only if $\overline{I}$ is an integral ideal.*

Proposition 19. *Let I and J be fractional ideals of an integral domain R. Then the following are equivalent:*

1. $I \leq J$.
2. $\overline{J} \subseteq \overline{I}$.
3. $J \subseteq \overline{I}$.

Proof. This is straightforward using the criterion of Proposition 8 part 3 and the fact that $J \subseteq \overline{J}$. □

Corollary 20. *Let R be an integral domain and let I and J be fractional ideals of R. Then $I \sim J$ if and only if $\overline{I} = \overline{J}$.*

Proposition 21. *Let I be a fractional ideal of an integral domain. Then $I \sim \overline{I}$. Thus $\overline{\overline{I}} = \overline{I}$. Moreover, $\overline{I}$ is the maximum element, under inclusion, in the collection of fractional ideals J such that $I \sim J$.*

Proof. We have $\overline{I} \leq I$ since $I \subseteq \overline{I}$ (Proposition 7). We also have $I \leq \overline{I}$ since $\overline{I} \subseteq \overline{I}$ (Proposition 19). So $I \sim \overline{I}$. Thus $\overline{\overline{I}} = \overline{I}$ (see above Corollary).

Suppose that $I \sim J$. Then $J \subseteq \overline{I}$ (Proposition 19). So $\overline{I}$ is the maximum such J. □

This allows us to regard $\overline{I}$ as the canonical representative of the equivalence class $[I]$:

Definition 8. Let I be a fractional ideal of an integral domain. The ideal $\overline{I}$ is called a *divisorial ideal*, and is called *the divisorial ideal associated with I*.

We can use formal inverses to characterize $\overline{I}$. In fact, as we will see $\overline{I}$ is just $\left(I^{-1}\right)^{-1}$.

Lemma 22. *Let I be a fractional ideal of an integral domain R. Then $I \leq \left(I^{-1}\right)^{-1}$.*

[2] These ideas are explored in more depth in Section 9.

Proof. We use the criterion of Proposition 8, part 2. So suppose $xI \subseteq R$ where x is in the fraction field K of R. Then $x \in I^{-1}$. Thus $x\left(I^{-1}\right)^{-1} \subseteq R$. □

Proposition 23. *Let I be a fractional ideal of an integral domain R. Then*

$$\overline{I} = \left(I^{-1}\right)^{-1}.$$

In particular, if I is an invertible fractional ideal then $\overline{I} = I$.

Proof. From Proposition 11 we have $I \subseteq \left(I^{-1}\right)^{-1}$. Thus $\left(I^{-1}\right)^{-1} \leq I$ (Proposition 7). When combined with the previous lemma, we get $I \sim \left(I^{-1}\right)^{-1}$. We just need to show that $\left(I^{-1}\right)^{-1}$ is the maximum in its equivalence class.

So suppose $I \sim J$. Then $J \subseteq \left(I^{-1}\right)^{-1}$ by Proposition 11. Thus $\left(I^{-1}\right)^{-1}$ is the maximum in the equivalence class of I, and must be $\overline{I}$. □

The above discussion is in the context of a general integral domain. Our goal, of course, is to have the identity $[I][I^{-1}] = [R]$. This will require more specialized rings. In particular, R should be Noetherian and integrally closed, or at least completely integrally closed (Definition 2).

Lemma 24. *Suppose R is an integrally closed Noetherian domain, or more generally a completely integrally closed domain. Then $II^{-1} \leq R$ for all fractional ideals I of R.*

Proof. We use the criterion of Proposition 8, part 2. Suppose $xII^{-1} \subseteq R$. This implies $xI^{-1} \subseteq I^{-1}$, so $x \in \mathcal{R}(I^{-1})$. By Proposition 3, $\mathcal{R}(I^{-1}) = R$. So $x \in R$. Thus we have $xR \subseteq R$. □

Theorem 25. *Suppose R is an integrally closed Noetherian domain, or more generally a completely integrally closed domain. Then*

$$[I]\left[I^{-1}\right] = [R]$$

for each fractional ideal I of R. So the divisor monoid of R is an Abelian group and the inverse of a divisor $[I]$ is just $\left[I^{-1}\right]$.

Proof. Since II^{-1} is an integral ideal we have $R \leq II^{-1}$ (Proposition 10). By the above lemma we have $II^{-1} \leq R$, and so $R \sim II^{-1}$. The result follows. □

There is a converse to this theorem. We begin with a lemma.

Lemma 26. *Suppose R is an integral domain whose divisor monoid satisfies the cancellation law. Suppose that K is the fraction field of R and that R' is a ring such that $R \subseteq R' \subseteq K$. If R' is a fractional ideal (as an R-submodule of K) then $R' = R$.*

Proof. If $I = R'$ then $II = I$ since R' is a ring. By the cancellation law, $[I] = [R]$. In particular, $R \leq I$. So $I \subseteq R$ (Proposition 10). In other words $R' \subseteq R$ and so $R' = R$. □

Theorem 27. *Suppose R is an integral domain whose divisor monoid is a group, or more generally whose divisor monoid satisfies the cancellation law. Then R is completely integrally closed.*

Proof. Let I be a fractional ideal of R. Then $\mathcal{R}(I)$ is a ring, and a fractional ideal containing R (Proposition 1). By the above lemma, $\mathcal{R}(I) = R$. Thus R is completely integrally closed (Proposition 3). $\square$

5 Irreducible and prime divisors

Our next goal is to identify irreducible and prime divisors of an integral domain R in terms of certain prime ideals of R.

Definition 9. An *integral divisor* of an integral domain R is a divisor of the form $[I]$ where I is a nonzero integral ideal.

In other words, if I is a nonzero integral ideal then $[I]$ is an integral divisor. The converse holds (see Proposition 14):

Lemma 28. *Let I be a fractional ideal of an integral domain. If $[I]$ is an integral divisor, then I must be an integral ideal.*

Proposition 29. *Let I be a fractional ideal of an integral domain R. Then $[I]$ is an integral divisor if and only if $[R] \leq [I]$.*

Corollary 30. *Let R be an integral domain. Then $[R]$ is the minimum integral divisor. In particular, if $[I] \leq [R]$ and $[I]$ is an integral divisor then $[I] = [R]$.*

Now we consider irreducible and prime divisors.

Definition 10. A integral divisor $[I]$ is said to be *irreducible* if (i) $[I] \neq [R]$ and (ii) if $[J] \leq [I]$ where $[J]$ is an integral divisor then $[J] = [R]$ or $[J] = [I]$.

Definition 11. A integral divisor $[P]$ is said to be *prime* if (i) $[P] \neq [R]$ and (ii) if $[P] \leq [I][J]$ where $[I]$ and $[J]$ are integral divisors then $[P] \leq [I]$ or $[P] \leq [J]$.

Proposition 31. *Let $[P]$ be an irreducible divisor in an integral domain R. Then $[P]$ is a prime divisor.*

Proof. The divisor $[P]$ integral and not equal to $[R]$ by assumption. Now suppose that $[P] \leq [I][J]$ where I, J are nonzero ideals of R. Then $IJ \subseteq \overline{P}$ (Proposition 19). This implies that $(I + \overline{P})(J + \overline{P}) \subseteq \overline{P}$. In particular $[P] \leq \left[I + \overline{P}\right]\left[J + \overline{P}\right]$.

Since $\overline{P} \subseteq I + \overline{P}$, we have $\left[I + \overline{P}\right] \leq [P]$ (Proposition 7). By irreducibility, either $\left[I + \overline{P}\right] = [R]$ or $\left[I + \overline{P}\right] = [P]$. In the second case, $P \leq I + \overline{P}$ so $I + \overline{P} \subseteq \overline{P}$ (Proposition 19). In this case $I \subseteq \overline{P}$, and $[P] \leq [I]$ as desired (Proposition 19).

Similarly, either $\left[J + \overline{P}\right] = [R]$ or $[P] \leq [J]$.

It cannot be that $\left[I + \overline{P}\right] = [R]$ and $\left[I + \overline{P}\right] = [R]$. Otherwise, $[P] \leq [R][R]$ which implies that $[P] = [R]$ (Corollary 30), a contradiction. So either $[P] \leq [I]$ or $[P] \leq [J]$. $\square$

Proposition 32. *Let $[P]$ be an integral divisor in an integral domain R. Then $[P]$ is a prime divisor if and only if $\overline{P}$ is a prime ideal.*

Proof. Suppose $[P]$ is a prime divisor. Since $[P] \neq [R]$, we have $\overline{P} \neq \overline{R} = R$. Thus $\overline{P}$ is a proper ideal. Suppose that $ab \in \overline{P}$. In other words $(aR)(bR) \subseteq \overline{P}$. Therefore $[P] \leq [aR][bR]$ (Proposition 19). Since $[P]$ is prime, we have two cases: either $[P] \leq [aR]$ or $[P] \leq [bR]$. In the first case, $aR \subseteq \overline{P}$ (Proposition 19), and $a \in \overline{P}$. In the second case $b \in \overline{P}$. So $\overline{P}$ is a prime ideal.

Now suppose $\overline{P}$ is a prime ideal. Since $\overline{P} \neq \overline{R} = R$, we have $[P] \neq [R]$. Suppose that $[P] \leq [I][J]$ where I, J are nonzero integral ideals. Thus $IJ \subseteq \overline{P}$ (Proposition 19). Since $\overline{P}$ is a prime ideal, either $I \subseteq \overline{P}$ or $J \subseteq \overline{P}$. In the first case, $[P] \leq [I]$, in the second case $[P] \leq [J]$ (Proposition 19). Thus $[P]$ is a prime divisor. □

The above is valid if R is a general integral domain where the divisor monoid is not necessarily a group. In the following we restrict ourselves to completely integrally closed domains in order to leverage the group properties.

Proposition 33. *Suppose I and J are fractional ideals in a completely integrally closed domain R. If $I \leq J$ then there are nonzero integral ideals I' and J' such that (i) $II' = JJ'$ and (ii) $[J'] = [R]$. In particular, $[I][I'] = [J]$.*

Proof. By Proposition 8 part 4 we have that $I' = JI^{-1}$ and $J' = II^{-1}$ are integral ideals. Also

$$II' = I(JI^{-1}) = J(II^{-1}) = JJ'.$$

Observe that $[J'] = [II^{-1}] = [R]$ since R is complete integrally closed (Theorem 25). In particular,

$$[I][I'] = [J][J'] = [J][R] = [J].$$

□

Proposition 34. *Suppose $[I]$ and $[J]$ are integral divisors in a completely integrally closed domain R. Then $[I] \leq [J]$ if and only if $[I]$ divides $[J]$ in the sense the there is an integral divisor $[I']$ such that $[I][I'] = [J]$.*

Proof. One implication follows from the previous proposition. If $[I][I'] = [J]$ then $II' \sim J$, so $II' \leq J$. Since $II' \subseteq I$ we have $I \leq II'$ (Proposition 7). Thus $I \leq J$ by transitivity of $\leq$. □

Proposition 35. *Suppose $[I]$ and $[J]$ are integral divisors in a completely integrally closed domain R. If $[I] = [J]$ then there are nonzero integral ideals I' and J' such that (i) $II' = JJ'$ and (ii) $[I'] = [J'] = [R]$.*

Proof. Let $I' = JI^{-1}$ and let $J' = II^{-1}$. Then

$$II' = I(JI^{-1}) = J(II^{-1}) = JJ'.$$

Now use Theorem 25 to conclude that $[I'] = [J'] = [R]$. Also I' and J' are integral ideals since $I \leq J$ and $I \leq I$ (Proposition 8 part 4). □

Now we continue our consideration of irreducible and prime divisors. We begin by expanding Proposition 31 to incorporate its converse:

Proposition 36. *Let $[P]$ be an integral divisor in a completely integrally closed domain R. Then $[P]$ is an irreducible divisor if and only if $[P]$ is a prime divisor.*

Proof. One direction has been shown (Proposition 31). So suppose $[P]$ is a prime divisor. Observe that $[P]$ is not equal to $[R]$ by the definition of prime divisor. Now suppose that $[I] \leq [P]$ where $[I]$ is an integral divisor. There is an integral divisor $[I']$ such that $[I][I'] = [P]$ (Proposition 34). Since $[P]$ is prime and since $[P] \leq [I][I']$, we have two cases: $[P] \leq [I]$ or $[P] \leq [I']$.

In the first case $[I] = [P]$ as desired for showing $[P]$ is irreducible.

In the second case $[P][I''] = [I']$ for some integral divisor $[I'']$ (Proposition 34). So $[I][P][I''] = [I][I'] = [P]$. After cancelling we get $[I][I''] = [R]$ (see Theorem 25). This means $[I] \leq [R]$ (Proposition 34). So $[I] = [R]$ (Corollary 30) as desired for showing $[P]$ is irreducible. $\square$

Now that we know that the irreducible divisors and prime divisors coincide, we will focus on the prime ideals associated to such divisors.

Proposition 37. *Suppose $\mathfrak{p}$ is a nonzero prime ideal in a completely integrally closed domain R. Then either (i) $\overline{\mathfrak{p}} = R$ and $[\mathfrak{p}] = [R]$, or (ii) $\overline{\mathfrak{p}} = \mathfrak{p}$ and $[\mathfrak{p}]$ is a prime divisor.*

Proof. Since $[\,\overline{\mathfrak{p}}\,] = [\mathfrak{p}]$ (Proposition 21) we have $\overline{\mathfrak{p}}I = \mathfrak{p}J$ for some nonzero integral ideals I and J with $[I] = [J] = [R]$ (Proposition 35). In particular, $\overline{\mathfrak{p}}I \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, either $I \subseteq \mathfrak{p}$ or $\overline{\mathfrak{p}} \subseteq \mathfrak{p}$.

In the first case $[\mathfrak{p}] \leq [I] = [R]$ (Proposition 7). So $[\mathfrak{p}] = [R]$ (Corollary 30), which implies $\overline{\mathfrak{p}} = R$ (Corollary 20).

In the second case, we also have $\mathfrak{p} \subseteq \overline{\mathfrak{p}}$ (Proposition 18) so $\overline{\mathfrak{p}} = \mathfrak{p}$. This implies also that $[\mathfrak{p}]$ is a prime divisor (Proposition 32). $\square$

The only type of prime ideal that can yield a prime divisor are minimal prime ideals.

Definition 12. Let R be an integral domain. Then a *minimal prime ideal* is a minimal element in the set of nonzero prime ideals. In other words, if $\mathfrak{p}$ is a minimal prime ideal and if $\mathfrak{q}$ is a nonzero prime ideal, then $\mathfrak{q} \subseteq \mathfrak{p}$ implies that $\mathfrak{q} = \mathfrak{p}$.

Proposition 38. *Suppose $\mathfrak{p}$ is a nonzero prime ideal in a completely integrally closed domain R. If $[\mathfrak{p}]$ is a prime divisor in R then $\mathfrak{p}$ is a minimal prime ideal of R.*

Proof. Suppose $\mathfrak{q}$ is a nonzero prime ideal with $\mathfrak{q} \subseteq \mathfrak{p}$. Then $[\mathfrak{p}] \leq [\mathfrak{q}]$ (Proposition 7). Observe that if $[\mathfrak{q}] = [R]$ then $[\mathfrak{p}] \leq [R]$, and so $[\mathfrak{p}] = [R]$ (Corollary 30), a contradiction since $[\mathfrak{p}]$ is assumed prime. So $[\mathfrak{q}] \neq [R]$, which by the above Proposition implies that $[\mathfrak{q}]$ is a prime divisor.

By the irreducibility of $[\mathfrak{q}]$ (Proposition 36) either $[\mathfrak{p}] = [R]$ or $[\mathfrak{p}] = [\mathfrak{q}]$. The first does not happen since $[\mathfrak{p}]$ is a prime divisor. Thus $[\mathfrak{p}] = [\mathfrak{q}]$ and hence $\overline{\mathfrak{p}} = \overline{\mathfrak{q}}$ (Corollary 20). But by the previous proposition, $\overline{\mathfrak{p}} = \mathfrak{p}$ and $\overline{\mathfrak{q}} = \mathfrak{q}$. Thus $\mathfrak{p} = \mathfrak{q}$. This show that $\mathfrak{p}$ is a minimal prime ideal. $\square$

Corollary 39. *Suppose $\mathfrak{p}$ is a nonzero prime ideal in a completely integrally closed domain R. If $\mathfrak{p}$ is not a minimal prime ideal of R then $[\mathfrak{p}] = [R]$.*

Proof. This is straightforward given Propositions 37 and 38. □

Corollary 40. *Suppose $[P]$ is a prime divisor in a completely integrally closed domain R. Then $\overline{P}$ is a minimal prime ideal of R.*

Proof. We have that $\mathfrak{p} = \overline{P}$ is a nonzero prime ideal by Proposition 32. Observe that $[P] = \left[\overline{P}\right] = [\mathfrak{p}]$ (Proposition 21). Since $[\mathfrak{p}]$ is a prime divisor, $\mathfrak{p}$ is a minimal prime ideal by Proposition 38. □

Question. Suppose $\mathfrak{p}$ is a minimal prime ideal in a completely integrally closed domain R. Is it always true that $[\mathfrak{p}]$ is a prime divisor, or is it possible for $[\mathfrak{p}]$ to be $[R]$? We will address this question in the next section at least for Krull domains.

In a general integral domain that is completely integrally closed we are not guaranteed the existence of a prime factorization for all divisors. But if a divisor has such a factorization, the factorization is essentially unique. This is assured by the following:

Proposition 41. *Suppose $[P_1], \ldots, [P_k]$ and $[Q_1], \ldots, [Q_l]$ are two finite sequences of prime divisors in a completely integrally closed domain R. If their respective products are equal,*

$$[P_1] \cdots [P_k] = [Q_1] \cdots [Q_l],$$

then $k = l$ and, after rearranging the order of the sequences, $[P_i] = [Q_i]$ for $1 \leq i \leq k$. We allow empty sequences, where the product of an empty sequence is considered to be the identity divisor $[R]$.

Proof. In the proof we can make the extra assumption that $k \geq l$. We prove the result by strong induction on $n = k + l$. If $n = 0$ then $k = l = 0$ and both sequences are the empty sequence. So the result holds.

Next we assume $n = k + l > 0$. Since $k \geq l$ we have $k > 0$. Suppose first that l is 0. In this case then $[P_k] \leq [R]$ (Proposition 34), so $[P_k] = [R]$ (Corollary 30). This cannot occur since $[P_k]$ is a prime divisor.

So we can assume both $k > 0$ and $l > 0$. Here $[P_k] \leq [Q_1] \cdots [Q_l]$ (Proposition 34). By definition of prime divisor, $[P_k] \leq [Q_i]$ for some i (Definition 11). After rearranging the $[Q_j]$ we can assume that $[P_k] \leq [Q_l]$. So $\overline{Q_l} \subseteq \overline{P_k}$ (Proposition 19). Since $\overline{Q_l}$ and $\overline{P_k}$ are both minimal prime ideals (Corollary 40), $\overline{P_k} = \overline{Q_l}$, which means $[P_k] = [Q_l]$ (Corollary 20). We can cancel (Theorem 25) obtaining

$$[P_1] \cdots [P_{k-1}] = [Q_1] \cdots [Q_{l-1}].$$

By the inductive hypothesis, $k - 1 = l - 1$ so $k = l$. Also by induction we can rearrange the divisors so that $[P_i] = [Q_i]$ for each i. □

6 Unique factorization of divisors in Krull domains

Now we will consider the topic of unique factorization of divisors. Above we worked either in the setting of an integral domain, where we have a monoid of divisors, or in the more specific setting of a completely integrally closed domain where we have a group of divisors.

In order to achieve unique factorization into prime divisors, we will need to go further and assume that our ring has some sort of ascending chain condition (ACC). An extra assumption that our ring is Noetherian would suffice but is actually stronger than what we need; we just need the ACC for divisorial integral ideals. This motivates our introduction of a kind of ring called a *Krull domain*:

Definition 13. A *Krull domain* is an integral domain R that satisfies the following properties:

1. R is completely integrally closed. (In other words $\mathcal{R}(I) = R$ for all fractional ideals I of R).

2. Every nonempty collection of divisorial integral ideals of R has a maximal element under inclusion. In other words, the ascending chain condition (ACC) holds for the collection of divisorial integral ideals of R. (Recall an ideal I is *divisorial* if and only if $\overline{I} = I$).

Remark. We could have expressed the second condition in terms of the DCC for the collection of integral divisors (relative to the relation $\le$).

Proposition 42. *Every integrally closed Noetherian domain R is a Krull domain. Hence every Dedekind domain, including any discrete valuation ring (DVR) or principal ideal domain (PID), is a Krull domain.*

Proof. Since R is an integrally closed Noetherian domain, it is automatically completely integrally closed (Proposition 4).

Every nonempty collection of ideals of a Noetherian ring has a maximal element by the ascending chain condition, so every nonempty collection of divisorial ideals must have a maximal element. □

Proposition 43. *Every integral divisor of a Krull domain R is the product of irreducible divisors, where the empty product is defined to be the identity divisor $[R]$.*

Proof. Suppose there are exceptions and let $\mathcal{S}$ be the set of nonzero integral ideals of the form $\overline{I}$ where $[I]$ is an exception. Since R is a Krull domain, there is a maximal element of $\mathcal{S}$. Let $\overline{I}$ be the maximal such element of $\mathcal{S}$, and let $[I]$ be the corresponding divisor.

Since $[I]$ is an exception, $[I]$ is not an irreducible divisor and $[I] \neq [R]$. Thus there is a divisor $[J]$ where $[J] \le [I]$ but where $[J] \neq [I]$ and $[J] \neq [R]$ (Definition 10). Also, $[I] = [J][J']$ for some divisor $[J']$ (Proposition 34). Thus $[J'] \le [I]$ (Proposition 34). Observe that $[J'] \neq [I]$; otherwise $[J][I] = [I]$, so by cancellation (Theorem 25) $[J] = [R]$, a contradiction.

Since $[J] \le [I]$ and $[J] \neq [I]$ we have $\overline{I} \subsetneq \overline{J}$ (Proposition 19 and Corollary 20). Similarly $\overline{I} \subsetneq \overline{J'}$. Thus $[J]$ and $[J']$ are not exceptions by definition of $\mathcal{S}$ and choice of $[I]$. This is a contradiction since $[I] = [J][J']$ would then be a product of irreducible divisors. □

Theorem 44. *Every integral divisor of a Krull domain R is uniquely the product of prime divisors, where the empty product is defined to be the identity divisor $[R]$ and where uniqueness is up to order.*

Proof. Existence of an irreducible factorization follows from the above proposition, and this must be a prime factorization (Proposition 31). Uniqueness was established in Proposition 41. $\square$

Krull domains are characterized by the unique factorization property for divisors:

Theorem 45. *Let R be an integral domain. Then R is a Krull domain if and only if every integral divisor of R is uniquely the product of prime divisors, where the empty product is defined to be the identity divisor $[R]$ and where uniqueness is up to order.*

Proof. One direction was established in the previous theorem. So suppose unique factorization of integral divisors holds. This yields a cancellation law for integral divisors. It is straightforward to show that if the cancellation law holds for integral divisors, it must hold for all divisors. Using Theorem 27 we conclude that R is completely integrally closed. So, according to the definition of Krull domain, we just need to establish the ACC for divisorial integral ideals.

By Proposition 41, the number of prime factors that occurs in the prime factorization of a given divisor is independent on the actual factorization. So if I is a nonzero integral ideal of R, let $n(I)$ be the number of prime factors in any factorization of the integral divisor $[I]$.

Suppose $I = \overline{I}$ and $J = \overline{J}$ are divisorial integral ideals such that $I \subsetneq J$. We will show that $n(J) < n(I)$. To do so, first observe that $[I] \neq [J]$ since $I \neq J$ and since I and J are divisorial (Corollary 20). Since $I \subsetneq J$ we have $J \leq I$ (Proposition 7). So there is an integral divisor $[J']$ such that $[J][J'] = [I]$ (Proposition 34). If $n(J') = 0$ then $[J'] = [R]$ which implies $[I] = [J]$, a contradiction. So $n(J') > 0$. Finally, since $[J][J'] = [I]$, we have $n(J) + n(J') = n(I)$ and so $n(J) < n(I)$ as desired.

To show that R is a Krull domain it remains to show that if $\mathcal{S}$ is a nonempty collection of divisorial integral ideals of R, there is a maximal element of $\mathcal{S}$ with respect to inclusion. Let $\mathcal{S}$ be such a collection, and let $I \in \mathcal{S}$ be any element such that $n(I)$ is as small as possible. If $J \in \mathcal{S}$ is such that $I \subsetneq J$ then $n(J) < n(I)$, which contradicts the choice of I. Thus I is a maximal element of $\mathcal{S}$. $\square$

We can apply factorization of divisors to complete the classification of prime divisors started in the previous section (see Proposition 37 and Corollaries 39 and 40):

Proposition 46. *Let $\mathfrak{p}$ be a minimal prime ideal of a Krull domain R. Then $[\mathfrak{p}]$ is a prime divisor and $\overline{\mathfrak{p}} = \mathfrak{p}$.*

Proof. Let $a \in \mathfrak{p}$ be a nonzero element and factor the divisor $[aR]$ into prime divisors (Theorem 44):

$$[aR] = [P_1] \cdots [P_k].$$

In particular, $aR \leq P_1 \cdots P_k$. If $\mathfrak{p}_i \stackrel{\text{def}}{=} \overline{P_i}$ then $\mathfrak{p}_i$ is a nonzero prime ideal (Proposition 32), and $\mathfrak{p}_i \sim P_i$ (Proposition 21). Thus $aR \leq \mathfrak{p}_1 \cdots \mathfrak{p}_k$ (Proposition 16). This implies that $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \overline{aR}$ (Proposition 19). Since aR is a principal ideal, $\overline{aR} = aR$ (Proposition 18). Thus

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \overline{aR} = aR \subseteq \mathfrak{p}.$$

Because $\mathfrak{p}$ is a prime ideal, $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. So $\mathfrak{p} = \mathfrak{p}_i$ since $\mathfrak{p}$ is a minimal prime ideal. So $[\mathfrak{p}] = [P_i]$ (Proposition 21) and is a prime divisor. Thus $\overline{\mathfrak{p}} = \mathfrak{p}$ (Proposition 37). □

We now can restate Theorem 44 as follows:

Theorem 47. *Let $[I]$ be an integral divisor of a Krull domain R. Then there is a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of distinct minimal prime ideals such that*

$$[I] = [\mathfrak{p}_1]^{n_1} \cdots [\mathfrak{p}_k]^{n_k}$$

For some $n_1, \ldots, n_k$ nonnegative integers. Given such a sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, the exponents $n_1, \ldots, n_k$ are unique.

We can extend this to general divisors:

Theorem 48. *Let $[I]$ be a divisor of a Krull domain R. Then there is a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of distinct minimal prime ideals such that*

$$[I] = [\mathfrak{p}_1]^{n_1} \cdots [\mathfrak{p}_k]^{n_k}$$

For some $n_1, \ldots, n_k \in \mathbb{Z}$. Given such a sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, the exponents $n_1, \ldots, n_k$ are unique.

Proof. If I is a fractional ideal, then I is of the form $(dR)^{-1}J$ for some nonzero integral ideals dR and J. This can be used to justify existence. Uniqueness is straightforward given Proposition 41. □

The sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ given in the proceeding theorems is not quite unique since we can make the following trivial modifications that do not affect the product: (1) we can add minimal prime ideals to the sequence with corresponding exponents zero, (2) similarly we can remove terms from the sequence if the corresponding exponents are zero, and (3) we can rearrange the order of the sequence. These are in some sense the only modifications:

Lemma 49. *Suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ are two sequences of distinct minimal prime ideals. Suppose*

$$[\mathfrak{p}_1]^{m_1} \cdots [\mathfrak{p}_k]^{m_k} = [\mathfrak{q}_1]^{n_1} \cdots [\mathfrak{q}_l]^{n_l}$$

where each m_i and n_i is an integer. Then the sequences differ (up to order) only in terms with exponent zero.

More precisely, if $\mathfrak{p}_i = \mathfrak{q}_j$ then $m_i = n_j$. Furthermore, a prime $\mathfrak{p}$ appears in the first sequences as some $\mathfrak{p}_i$ with nonzero exponent $m_i \neq 0$ if and only if it appears in the second sequence as some $\mathfrak{q}_j$ with nonzero exponent $n_j \neq 0$, and vice versa.

Proof. The idea is to use, if necessary, a third sequence of distinct minimal prime ideals that incorporates every $\mathfrak{p}_i$ and $\mathfrak{q}_i$. Then use the uniqueness of exponents for this third sequence. □

7 Discrete valuations associated to irreducible divisors

We begin with some additional basic facts that hold in the context of a general integral domain R:

Proposition 50. *Let aR and bR be nonzero principal ideals of an integral domain R. Then $aR \leq bR$ if and only if a divides b in R.*

Proposition 51. *Let xR and yR be principal fractional ideals of an integral domain R. Then $[xR] = [yR]$ if and only if $xR = yR$.*

Proposition 52. *Let I_1 and I_2 be fractional ideals of an integral domain. If J is a fractional ideal then $J \leq I_1$ and $J \leq I_2$ if and only if $J \leq I_1 + I_2$.*

Proof. If $J \leq I_1$ and $J \leq I_2$ then $I_1 \subseteq \overline{J}$ and $I_2 \subseteq \overline{J}$ (Proposition 19). This implies that $I_1 + I_2 \subseteq \overline{J}$. Thus $J \leq I_1 + I_2$ (Proposition 19).

The converse follows from the following observation: since $I_i \subseteq I_1 + I_2$ we have $I_1 + I_2 \leq I_i$. Thus if $J \leq I_1 + I_2$ then $J \leq I_1$ and $J \leq I_2$ by transitivity. □

Definition 14. Let $\mathfrak{p}$ be a minimal prime ideal of a Krull domain R. Suppose I is a fractional ideal of R with divisor factorization

$$[I] = [\mathfrak{p}_1]^{n_1} \cdots [\mathfrak{p}_k]^{n_k}.$$

Here $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ is a sequence of distinct minimal ideals that we assume includes $\mathfrak{p}$, as $\mathfrak{p}_i$, say. We define the *valuation of I at $\mathfrak{p}$* as follows:

$$v_{\mathfrak{p}}(I) \stackrel{\text{def}}{=} n_i.$$

This is well-defined by Theorem 48 and Lemma 49.

Proposition 53. *Let I and J be fractional ideals of a Krull domain R, and let $\mathfrak{p}$ be a minimal prime ideal of R. Then*

$$v_{\mathfrak{p}}(IJ) = v_{\mathfrak{p}}(I) + v_{\mathfrak{p}}(J).$$

Proposition 54. *Let I be a fractional ideals of a Krull domain R. Then I is an integral ideal if and only if $v_{\mathfrak{p}}(I) \geq 0$ for all minimal prime ideals $\mathfrak{p}$ of R.*

Proposition 55. *Let I be a fractional ideals of a Krull domain R. Then $v_{\mathfrak{p}}(I) \neq 0$ for only finitely many minimal prime ideals $\mathfrak{p}$ of R.*

Proposition 56. *Let I, J be fractional ideals of a Krull domain R. Then $[I] = [J]$ if and only if $v_{\mathfrak{p}}(I) = v_{\mathfrak{p}}(J)$ for all minimal prime ideals $\mathfrak{p}$ of R.*

Proposition 57. *Let I, J be fractional ideals of a Krull domain R. Then $I \leq J$ if and only if $v_{\mathfrak{p}}(I) \leq v_{\mathfrak{p}}(J)$ for all minimal prime ideals $\mathfrak{p}$ of R.*

Proof. If $I \leq J$ then $[I][I'] = [J]$ for some integral ideal I' (Proposition 33). This means $v_{\mathfrak{p}}(I) + v_{\mathfrak{p}}(I') = v_{\mathfrak{p}}(J)$ (Proposition 53), and the result follows (Proposition 54).

Conversely suppose $v_{\mathfrak{p}}(I) \leq v_{\mathfrak{p}}(J)$ for all minimal prime ideals $\mathfrak{p}$ of R. We can replace I and J with equivalent ideals (Propositions 16 and 56). In particular, we can replace I and J with products of powers of minimal prime ideals. In that case we get $II' = J$ at the level of fractional ideals for some integral ideal I'. So $J \subseteq I$ and thus $I \leq J$ (Proposition 7). $\square$

Proposition 58. *Let I_1, I_2 be fractional ideals of a Krull domain R. Then*

$$v_\mathfrak{p}(I_1 + I_2) = \min\{v_\mathfrak{p}(I_1), v_\mathfrak{p}(I_2)\}$$

for all minimal prime ideals $\mathfrak{p}$ of R.

Proof. Since $I_1 \subseteq I_1 + I_2$ we have $I_1 + I_2 \leq I_1$ (Proposition 7). So $v_\mathfrak{p}(I_1 + I_2) \leq v_\mathfrak{p}(I_1)$ for each minimal prime ideal $\mathfrak{p}$ (previous proposition). Similarly, for all minimal prime ideals $\mathfrak{p}$ of R we have $v_\mathfrak{p}(I_1 + I_2) \leq v_\mathfrak{p}(I_2)$ so

$$v_\mathfrak{p}(I_1 + I_2) \leq \min\{v_\mathfrak{p}(I_1), v_\mathfrak{p}(I_2)\}.$$

By using products of minimal prime ideals, we can produce a fractional ideal J such that $v_\mathfrak{p}(J) = \min\{v_\mathfrak{p}(I_1), v_\mathfrak{p}(I_2)\}$ for all minimal prime ideals $\mathfrak{p}$ of R. By the above proposition we have $J \leq I_1$ and $J \leq I_2$. So by Proposition 52, $J \leq I_1 + I_2$. Thus (by the previous proposition) for all minimal prime ideals $\mathfrak{p}$ of R

$$\min\{v_\mathfrak{p}(I_1), v_\mathfrak{p}(I_2)\} = v_\mathfrak{p}(J) \leq v_\mathfrak{p}(I_1 + I_2).$$

$\square$

Proposition 59. *Let I be a nonzero integral ideal of a Krull domain R and let $\mathfrak{p}_0$ be a minimal prime ideal of R. Then the following are equivalent:*

1. *$I \subseteq \mathfrak{p}_0$*
2. *$\mathfrak{p}_0 \leq I$*
3. *$v_{\mathfrak{p}_0}(I) \geq 1$.*

Proof. The implication (1) $\implies$ (2) follows from Proposition 7. Suppose $\mathfrak{p}_0 \leq I$. Then $I \subseteq \overline{\mathfrak{p}_0}$ (Proposition 19) and $\overline{\mathfrak{p}_0} = \mathfrak{p}_0$ (Proposition 46). So (2) $\implies$ (1).

If $\mathfrak{p}_0 \leq I$ then $1 = v_{\mathfrak{p}_0}(\mathfrak{p}_0) \leq v_{\mathfrak{p}_0}(I)$ (Proposition 57), so (2) $\implies$ (3).

If $v_{\mathfrak{p}_0}(I) \geq 1$ then $v_\mathfrak{p}(\mathfrak{p}_0) \leq v_\mathfrak{p}(I)$ for all minimal prime ideals $\mathfrak{p}$ of R. This implies $\mathfrak{p}_0 \leq I$ (Proposition 57), so (3) $\implies$ (2). $\square$

Corollary 60. *Suppose $a \in R$ is a nonzero element of a Krull domain R. If a is not a unit of R then $a \in \mathfrak{p}$ for some minimal prime ideal $\mathfrak{p}$ of R.*

Proof. Since a is not a unit, $aR \neq R$. So $[aR] \neq [R]$ (Proposition 51). This means that $v_\mathfrak{p}(aR) \geq 1$ for some minimal prime ideal $\mathfrak{p}$ of R (Proposition 54 and Proposition 56). Thus $aR \subseteq \mathfrak{p}$ by the above proposition. So $a \in \mathfrak{p}$ as desired. $\square$

Corollary 61. *Suppose $a \in R$ is a nonzero element of a Krull domain R. Then a is contained in only a finite number of minimal prime ideals.*

Proof. Suppose that $a \in \mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime ideal. So $aR \subseteq \mathfrak{p}$ and by the above proposition $v_\mathfrak{p}(aR) \geq 1$. There can only be a finite number of such minimal prime ideals $\mathfrak{p}$ (Proposition 55). $\square$

Now we consider valuations at the level of elements.

Definition 15. Let R be an Krull domain with field of fractions K. If $x \in K^\times$ and if $\mathfrak{p}$ is a minimal prime of R then

$$v_\mathfrak{p}(x) \stackrel{\text{def}}{=} v_\mathfrak{p}(xR).$$

When needed, we define $v_\mathfrak{p}(0)$ to be ∞.

Lemma 62. *Let R be a Krull domain and let $\mathfrak{p}$ be a minimal prime of R. Then there is an element $\pi \in R$ such that $v_\mathfrak{p}(\pi) = 1$.*

Proof. Since $[\mathfrak{p}] \neq [R]$ (Proposition 46) and since the divisors form a group (Theorem 25), we have $[\mathfrak{p}^2] \neq [\mathfrak{p}]$. In particular, $\overline{\mathfrak{p}^2} \neq \overline{\mathfrak{p}}$ (Corollary 20).

Since $\mathfrak{p}^2 \subseteq \mathfrak{p}$ we have $[\mathfrak{p}] \leq [\mathfrak{p}^2]$ (Proposition 7) so $\overline{\mathfrak{p}^2} \subseteq \overline{\mathfrak{p}}$ (Proposition 19). Since $\overline{\mathfrak{p}^2} \subsetneq \overline{\mathfrak{p}}$, there is an element $\pi \in \overline{\mathfrak{p}} \smallsetminus \overline{\mathfrak{p}^2}$.

Since $\pi R \subseteq \overline{\mathfrak{p}}$ we have $\mathfrak{p} \leq \pi R$ (Proposition 19). So by Proposition 57

$$1 = v_\mathfrak{p}(\mathfrak{p}) \leq v_\mathfrak{p}(\pi R) = v_\mathfrak{p}(\pi).$$

Suppose that $v_\mathfrak{p}(\pi) \geq 2$. In this case $v_\mathfrak{q}(\mathfrak{p}^2) \leq v_\mathfrak{q}(\pi R)$ for all minimal primes $\mathfrak{q}$ of R, which means that $\mathfrak{p}^2 \leq aR$ (Proposition 57). Thus $\pi R \subseteq \overline{\mathfrak{p}^2}$ (Proposition 19). This means $\pi \in \overline{\mathfrak{p}^2}$ which contradicts the choice of π. We conclude that $v_\mathfrak{p}(\pi) = 1$ as desired. □

Proposition 63. *Let R be a Krull domain with field of fractions K. Let $\mathfrak{p}$ be a minimal prime of R. Then the function $v_\mathfrak{p} \colon K^\times \to \mathbb{Z}$ is a discrete valuation of K.*

Proof. The identity

$$v_\mathfrak{p}(xy) = v_\mathfrak{p}(x) + v_\mathfrak{p}(y)$$

follows from Proposition 53. Suppose $x, y \in K^\times$ are such that $x + y \neq 0$. Then we have the inclusion $(x+y)R \subseteq xR+yR$, and so $xR+yR \leq (x+y)R$ (Proposition 7). In particular, $v_\mathfrak{p}(x+y) \geq v_\mathfrak{p}(xR+yR)$ (Proposition 57). By Proposition 58,

$$v_\mathfrak{p}(xR+yR) = \min\{v_\mathfrak{p}(xR), v_\mathfrak{p}(yR)\},$$

so

$$v_\mathfrak{p}(x+y) \geq \min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\}.$$

Finally we need to show that the function $v_\mathfrak{p} \colon K^\times \to \mathbb{Z}$ is surjective. Observe that the image is a subgroup of $\mathbb{Z}$ since, as established above, $v_\mathfrak{p}$ is a group homomorphism. By Lemma 62, 1 is in the image. Thus the image of $v_\mathfrak{p}$ is all of $\mathbb{Z}$. □

From Proposition 54 we get the following:

Proposition 64. *Let a be a nonzero element of the fraction field of a Krull domain R. Then $a \in R$ if and only if $v_\mathfrak{p}(a) \geq 0$ for all minimal primes $\mathfrak{p}$ of R.*

Proposition 65. *Let a be a nonzero element of a Krull domain R and let $\mathfrak{p}$ be a minimal prime ideal of R. Then $v_\mathfrak{p}(a) \geq 1$ if and only if $a \in \mathfrak{p}$.*

Proof. This follows from Proposition 59. □

Proposition 66. *Let x be a nonzero element of the function field of a Krull domain R. Then $v_{\mathfrak{p}}(x) \neq 0$ for only a finite number of minimal prime ideals $\mathfrak{p}$ of R.*

Proof. Write x as a/b with $a, b \in R$. So $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b)$ for all minimal prime ideals $\mathfrak{p}$ of R. The result follows from Corollary 61, Proposition 64, and Proposition 65 applied to a and b. $\square$

8 Localization at minimal primes in Krull domains

Let R be a Krull domain. Let $\mathfrak{p}$ be a minimal prime ideal of R and let $v_{\mathfrak{p}}$ be the valuation associated with $\mathfrak{p}$. Recall that every ideal of the localization $R_{\mathfrak{p}}$ is of the form $IR_{\mathfrak{p}}$ where I is an ideal of R. Let $IR_{\mathfrak{p}}$ be such an ideal, and assume it is not the zero ideal. By the unique factorization (Theorem 48) we can write

$$[I] = [\mathfrak{p}_1]^{n_1} \cdots [\mathfrak{p}_k]^{n_k}$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ is a sequence of distinct minimal prime ideals of R and where each $n_i \geq 0$. We can assume that the sequence is chosen so that $\mathfrak{p}_1$ is $\mathfrak{p}$. We can expand this factorization of divisors into a factorization of ideals (see Proposition 35):

$$II' = \mathfrak{p}_1{}^{n_1} \cdots \mathfrak{p}_k{}^{n_k} J'$$

where $I' \sim J' \sim R$. Note that, $v_{\mathfrak{p}}(I') = v_{\mathfrak{p}}(J') = v_{\mathfrak{p}}(R) = 0$. Also $v_{\mathfrak{p}}(\mathfrak{p}_i) = 0$ unless the index i is 1.

In general, if J is a nonzero ideal of R such that $v_{\mathfrak{p}}(J) = 0$ then J is not a subset of $\mathfrak{p}$ (Proposition 59); for such J there is an element $s \in J$ where $s \notin \mathfrak{p}$, and so $JR_{\mathfrak{p}}$ is the identity ideal $R_{\mathfrak{p}}$. In particular, from

$$(IR_{\mathfrak{p}})(I'R_{\mathfrak{p}}) = (\mathfrak{p}_1 R_{\mathfrak{p}})^{n_1} \cdots (\mathfrak{p}_k R_{\mathfrak{p}})^{n_k} (J'R_{\mathfrak{p}})$$

we get

$$(IR_{\mathfrak{p}})R_{\mathfrak{p}} = (\mathfrak{p}R_{\mathfrak{p}})^{n_1} (R_{\mathfrak{p}})^{n_2} \cdots (R_{\mathfrak{p}})^{n_k} R_{\mathfrak{p}}.$$

Thus $IR_{\mathfrak{p}} = (\mathfrak{p}R_{\mathfrak{p}})^{n_1}$. Observe that $n_1 = v_{\mathfrak{p}}(I)$. So we have proved the following:

Lemma 67. *Let R be a Krull domain and let $\mathfrak{p}$ be a minimal prime ideal of R. If I is a nonzero ideal of R then*

$$IR_{\mathfrak{p}} = (\mathfrak{p}R_{\mathfrak{p}})^{v_{\mathfrak{p}}(I)}.$$

In particular, every nonzero ideal of the localization $R_{\mathfrak{p}}$ is of the form $(\mathfrak{p}R_{\mathfrak{p}})^n$ for some $n \geq 0$.

Theorem 68. *Let R be a Krull domain and let $\mathfrak{p}$ be a minimal prime ideal of R. Then $R_{\mathfrak{p}}$ is a discrete valuation ring (DVR).*

Proof. Let $\pi \in R$ be such that $v_{\mathfrak{p}}(\pi) = 1$ (Lemma 62). By the above lemma, $\pi R_{\mathfrak{p}}$ is equal to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. The maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ is principal, which implies that $(\mathfrak{p}R_{\mathfrak{p}})^n$ is principal for all $n \geq 0$. By the above lemma, every ideal of $R_{\mathfrak{p}}$ is principal.

Since R is a local PID, it must be a DVR. $\square$

Proposition 69. *Let R be a Krull domain with field of fractions K. Let $\mathfrak{p}$ be a minimal prime ideal of R. Then the valuation ring of $v_{\mathfrak{p}}$ is the localization $R_{\mathfrak{p}}$ of R at the prime ideal $\mathfrak{p}$.*

Proof. We begin by showing that $R_{\mathfrak{p}}$ is contained in $\mathcal{O}_{v_{\mathfrak{p}}}$ where $\mathcal{O}_{v_{\mathfrak{p}}}$ is the valuation ring of $v_{\mathfrak{p}}$.

If $a \in R$ is nonzero then $v_{\mathfrak{p}}(a) \geq 0$ (Proposition 64) and $v_{\mathfrak{p}}(a) \geq 1$ if and only if $a \in \mathfrak{p}$ (Proposition 65). So if $s \in R \smallsetminus \mathfrak{p}$ then $v_{\mathfrak{p}}(s) = 0$. Suppose $a/s \in R_{\mathfrak{p}}$ with $a, s \in R$ and $s \notin \mathfrak{p}$ then

$$v_{\mathfrak{p}}(a/s) = v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(s) = v_{\mathfrak{p}}(a) - 0 \geq 0.$$

So a/s is in the valuation ring $O_{v_{\mathfrak{p}}}$.

This gives $R_{\mathfrak{p}} \subseteq \mathcal{O}_{v_{\mathfrak{p}}}$. Equality now follows from the fact that $R_{\mathfrak{p}}$ is a DVR. (This is a general principal, but there is an argument adapted to the current situation. Since it is short we give it here. Suppose $x \in \mathcal{O}_{v_{\mathfrak{p}}}$ is nonzero. Either x or its multiplicative inverse y are in $R_{\mathfrak{p}}$ since $R_{\mathfrak{p}}$ is a DVR. If $y \in R_{\mathfrak{p}}$ then $y \in \mathcal{O}_{v_{\mathfrak{p}}}$ by the above inclusion. So y is a unit of $\mathcal{O}_{v_{\mathfrak{p}}}$. Thus $v_{\mathfrak{p}}(y) = 0$, which implies that $y \notin \mathfrak{p}R_{\mathfrak{p}}$ so y is a unit of the local ring $R_{\mathfrak{p}}$. Thus $x \in R_{\mathfrak{p}}$ if $y \in R_{\mathfrak{p}}$. So in any case $x \in R_{\mathfrak{p}}$.) ☐

Corollary 70. *Let R be a Krull domain with fraction field K. Suppose R is not all of K. Then*

$$R = \bigcap_{\mathfrak{p}\ \mathit{minimal}} R_{\mathfrak{p}}$$

where the intersection is indexed by the set of minimal prime ideals of R.

Proof. One inclusion is automatic, so let a be a nonzero element in the right hand side. Then $v_{\mathfrak{p}}(a) \geq 0$ for all minimal primes $\mathfrak{p}$ of R. Hence $a \in R$ (Proposition 64). ☐

9 Bounds for the divisibility relation

We start with a fairly abstract but simple situation. Let X be a set equipped with a relation $\leq$. Given a subset $S \subseteq X$, define the lower bound set $L(S)$ and upper bound set $U(S)$ as follows:

$$L(S) = \{x \in X \mid x \leq s \text{ for all } s \in S\}, \quad U(S) = \{x \in X \mid s \leq x \text{ for all } s \in S\}.$$

Lemma 71. *Let S be a subset of X. Then $S \subseteq L(U(S))$ and $S \subseteq U(L(S))$.*

Proof. Let $a \in S$. Suppose $b \in U(S)$. Then $a \leq b$ by definition of $U(S)$. This holds for all $b \in U(S)$. So $a \in L(U(S))$. A similar arguments shows $S \subseteq L(U(S))$. ☐

We observe that L and U are order reversing:

Lemma 72. *Let S and T be subsets of X. If $S \subseteq T$ then*

$$L(T) \subseteq L(S) \quad \textit{and} \quad U(T) \subseteq U(S).$$

Lemma 73. *Let S bs a subset of X. Then*

$$L(S) = L(U(L(S))) \quad \textit{and} \quad U(S) = U(L(U(S))).$$

Proof. We prove the first statement. The second is similar.

The inclusion $L(S) \subseteq L(U(L(S)))$ is a special case of Lemma 71. Also by Lemma 71, $S \subseteq U(L(S))$. Thus $L(U(L(S))) \subseteq L(S)$ by Lemma 72. $\square$

We can use either lower bounds or upper bounds to define an induced relation on the power set of X. Here we will just consider the induced relation associated to lower bounds since that is what is used in the theory of divisors: if S and T are subsets of X, then $S \leq T$ is defined to mean that $L(S) \subseteq L(T)$. In this situation we define the closure of S as follows:

$$\overline{S} \stackrel{\text{def}}{=} U(L(S)).$$

We note that $S \subseteq \overline{S}$ and $\overline{S} = \overline{\overline{S}}$ holds for all subsets S of X. We call sets of the form $\overline{S}$ *closed subsets* of X, and we call $\overline{S}$ the *closure of* S. We note that a subset S of X is closed if and only if it is of the form $U(T)$ for some subset T. Also, the following are equivalent by the above lemmas:

1. $S \leq T$, in other words $L(S) \subseteq L(T)$.
2. $\overline{T} \subseteq \overline{S}$.
3. $T \subseteq \overline{S}$.

Now we specialize this to the situation that we really care about, the divisibility relation in an integral domain. Let R be an integral domain and let K be the fraction field of R. We specialize to the case where $X = K$ and where $\leq$ is divisibility: $x \leq y$ means $y = ax$ for some $a \in R$.

Lemma 74. *Suppose $x, y \in K$. Then the following are equivalent:*

1. *$x \leq y$.*
2. *$y \in xR$.*
3. *$yR \subseteq xR$.*

Observe that $x \leq 0$ for all $x \in K$, so $0 \in U(S)$ for all S. Recall that a subset of $X = K$ is closed if and only if it is of the form $U(S)$, so 0 is an element of any closed set. Observe that $U(\varnothing) = K$. If $R \neq K$ then K is not a fractional ideal, so the following lemma shows that $U(S) = K$ only when $S = \varnothing$.

Lemma 75. *Let $S \subseteq K$ be nonempty. Then*

$$U(S) = \bigcap_{x \in S} xR.$$

In particular, $U(S)$ is a R-submodule of K, and $U(S)$ is either the zero module or a fractional ideal of R.

Lemma 76. *Let S be a subset of K. Suppose $x \in K$, then $x \in L(S)$ if and only if $S \subseteq xR$. In particular, if $L(S)$ is nonempty then*

$$\overline{S} = U(L(S)) = \bigcap_{x \in L(S)} xR = \bigcap_{S \subseteq xR} xR$$

A consequence of the first assertion of the above lemma is that an R-submodule I of K is a fractional ideal if and only if $L(I)$ is nonempty and does not contain 0. In particular, the above formula for $\overline{S}$ applies to fractional ideals $S = I$ and agrees with the earlier definition of $\overline{I}$ (Definition 7). We can also use these ideas to check that the closure of a fractional ideal is a fractional ideal (as in Proposition 18): if I is a fractional ideal then $L(I)$ is nonempty and does not contain 0, and since

$$L\left(\overline{I}\right) = L(U(L(I))) = L(I)$$

the same is true of $L(\overline{I})$. So $\overline{I}$ must also be a fractional ideal. Recall that the closure of a fractional ideal is called a *divisorial ideal* (Definition 8). In other words, a divisorial ideal is a closed fractional ideal.

Now we go back to the abstract setting of a relation $\leq$ on a set X. We suppose that there is a second relation $\leq'$ on X such that the following holds for all $x, y \in X$:

$$\text{if } x \leq y \text{ then } x \leq' y.$$

For each subset S of X we define $L'(S)$ and $U'(S)$ using the relation $\leq'$. Observe that $L(S) \subseteq L'(S)$ and $U(S) \subseteq U'(S)$ for any subset S of X.

Lemma 77. *Let S and T be subsets of X. If $S \subseteq T$ then $U'(L(S)) \subseteq U'(L(T))$.*

For closed subsets we get the following:

Lemma 78. *If S is a subset of X then let $\Phi(S)$ be $U'(L(S))$. Then $S \subseteq \overline{S} \subseteq \Phi(S)$ where $\overline{S} = U(L(S))$ is the closure with respect to $\leq$.*

The function Φ maps the collection of closed subsets of X (under $\leq$) to the collection of closed subsets of X (under $\leq'$). This function Φ is order preserving: if $S \subseteq T$ then $\Phi(S) \subseteq \Phi(T)$, or equivalently if $T \leq S$ then $\Phi(T) \leq' \Phi(S)$.

An important example of the above situation is where R and R' are integral domains with common fraction field K and where $R \subseteq R'$. In this situation $X = K$, the first relation $\leq$ is the divisibility relation for R, and the second relation $\leq'$ is the divisibility relation for R'. Observe that this situation conforms to the requirement that $x \leq y$ implies $x \leq' y$ since $y \in xR$ implies $y \in xR'$.

Lemma 79. *Let R and R' be integral domains with common fraction field K. Assume $R \subseteq R'$. If I is a divisorial ideal of R then let $\Phi(I)$ be $U'(L(I))$. Then $\Phi(I)$ is a divisorial ideal of R'.*

Proof. By Lemma 78, $\Phi(I)$ is closed with respect to $\leq'$. Since I is a fractional ideal, $L(I)$ is nonempty. So $\Phi(I)$ is either the zero module or a fractional ideal (Lemma 75). But $I \subseteq \Phi(I)$ (Lemma 78), so $\Phi(I)$ is not the zero ideal and so must be a divisorial ideal. $\square$

The map behaves as expected for principal ideals:

Lemma 80. *Let R and R' be integral domains with common fraction field K. Assume $R \subseteq R'$. If $x \in K^\times$ then*

$$\Phi(xR) \stackrel{\text{def}}{=} U'(L(xR)) = xR'.$$

Proof. Note that $x \in L(xR)$. So if $y \in U'(L(xR))$ then $x \leq' y$. In other words, if $y \in \Phi(xR)$ then $y \in xR'$, so $\Phi(xR) \subseteq xR'$. On the other hand, $xR \subseteq \Phi(xR)$ (Lemma 78). So $x \in \Phi(xR)$, and since $\Phi(xR)$ is a fractional ideal this implies the other inclusion $xR' \subseteq \Phi(xR)$. □

Corollary 81. *If I is an integral divisorial ideal then $\Phi(I) = U'(L(I))$ is an integral divisorial divisor of R'.*

Proof. If I is an integral divisorial ideal then $I \subseteq R$. So $\Phi(I) \subseteq \Phi(R) = R'$. Now use Lemma 79. □

Again we return to the abstract setting of a relation $\leq$ on a set X. This time we suppose that there is also a nonempty family of relations $(\leq_i)_{i\in\mathcal{I}}$ on X such that the following holds for all $x, y \in X$:

$$x \leq y \text{ if and only if } x \leq_i y \text{ for each } i \in \mathcal{I}.$$

For each $i \in \mathcal{I}$ and each subset S of X we define $L_i(S)$ and $U_i(S)$ using the relation $\leq_i$. Observe that

$$L(S) = \bigcap_{i\in\mathcal{I}} L_i(S), \quad \text{and} \quad U(S) = \bigcap_{i\in\mathcal{I}} U_i(S).$$

Lemma 82. *Let $\overline{S} = S$ and $\overline{T} = T$ be closed subsets of X. Then $S \subseteq T$ if and only*

$$U_i(L(S)) \subseteq U_i(L(T))$$

for each $i \in \mathcal{I}$.

Proof. If $S \subseteq T$ then $L(T) \subseteq L(S)$. Thus $U_i(L(S)) \subseteq U_i(L(T))$ for each $i \in \mathcal{I}$.

Conversely if $U_i(L(S)) \subseteq U_i(L(T))$ for each $i \in \mathcal{I}$, then by taking the intersection we get

$$S = \overline{S} = U(L(S)) = \bigcap_{i\in\mathcal{I}} U_i(L(S)) \subseteq \bigcap_{i\in\mathcal{I}} U_i(L(T)) = U(L(T)) = \overline{T} = T.$$

□

We apply this result to the situation where R is an integral domain with fraction field K such that $R = \bigcap R_i$ where $(R_i)_{i\in\mathcal{I}}$ is a nonempty family of subrings of K. In this situation, let $X = K$, let $\leq$ be the divisibility relation for R: $x \leq y$ if and only if $y \in xR$, and for each $i \in \mathcal{I}$ let $\leq_i$ be the divisibility relation for R_i: $x \leq_i y$ if and only if $y \in xR_i$. Observe the following for each $x, y \in K$:

$$x \leq y \text{ if and only if } x \leq_i y \text{ for each } i \in \mathcal{I}.$$

For each $i \in \mathcal{I}$ we have, as before, an order preserving function Φ_i from the set of divisorial ideals of R to the set of divisorial ideals of R_i given by the rule

$$\Phi_i(I) \stackrel{\text{def}}{=} U_i(L(I)).$$

This map sends integral divisorial ideals of R to integral divisorial ideals of R_i (Corollary 81). In the current situation, we can rephrase Lemma 82:

Lemma 83. *Let I and J be divisorial ideals of R. Then $I \subseteq J$ if and only*

$$\Phi_i(I) \subseteq \Phi_i(J)$$

for each $i \in \mathcal{I}$.

Remark. By switching order, we can rephrase the above in terms of the $\leq$ relation on divisorial ideals: $J \leq I$ if and only if $\Phi_i(J) \leq \Phi_i(I)$ in R_i for each $i \in \mathcal{I}$.

The main payoff for the constructions in this section is the following:

Proposition 84. *Let R be an integral domain with fraction field K. Let $(R_i)_{i\in\mathcal{I}}$ be a nonempty family of subrings of K such that (i)*

$$R = \bigcap_{i\in\mathcal{I}} R_i,$$

(ii) for every nonzero $a \in R$ there are at most a finite number of $i \in \mathcal{I}$ such that a is not a unit of R_i, and (iii) the ACC holds for the collection of integral divisorial ideals of R_i for each $i \in \mathcal{I}$. Then the ACC holds for the collection of integral divisorial ideals of R.

Proof. Consider an infinite ascending chain of integral divisorial ideals of R

$$I_u \subseteq I_{u+1} \subseteq I_{u+2} \subseteq \cdots .$$

Our goal is to show that this chain stabilizes. Let $a \in I_u$ be nonzero. Then aR is an integral divisorial ideal and $aR \subseteq I_j$ for all I_j in the chain. By assumption there is a finite subset $\mathcal{I}_0$ of $\mathcal{I}$ such that for $i \notin \mathcal{I}_0$ the element a is a unit in R_i, which means $aR_i = R_i$ when $i \notin \mathcal{I}_0$.

For each $i \in \mathcal{I}$, let Φ_i be as above. For each $i \notin \mathcal{I}_0$ and each I_j we have that $aR \subseteq I_j \subseteq R$ and $aR_i = R_i$. So by Lemma 80 and Lemma 78,

$$R_i = aR_i = \Phi_i(aR) \subseteq \Phi(I_j) \subseteq \Phi_i(R) = R_i.$$

Thus for $i \notin \mathcal{I}_0$ the chain $(\Phi_i(I_j))$ is the constant chain. We set $N_i = u$ in this case.

For each index i in the finite set $\mathcal{I}_0$, we get a chain of integral divisorial ideal of R_i (Lemma 78, Corollary 81):

$$\Phi_i(I_u) \subseteq \Phi_i(I_{u+1}) \subseteq \Phi_i(I_{u+2}) \subseteq \cdots .$$

By the ACC assumption for R_i, this chain stabilizes and so is constant for $j \geq N_i$ for some index N_i.

Let N be the maximum of the N_i with $i \in \mathcal{I}$. If $j \geq N$ then $\Phi_i(I_j) = \Phi_i(I_{j+1})$ for all $i \in \mathcal{I}$. Thus $I_j = I_{j+1}$ by Lemma 83. This shows that the chain (I_j) stabilizes, confirming the ACC for integral divisorial ideals. $\square$

Corollary 85. *Let R be an integral domain with fraction field K. Let $(R_i)_{i\in\mathcal{I}}$ be a nonempty family of subrings of K such that (i)*

$$R = \bigcap_{i\in\mathcal{I}} R_i,$$

(ii) for every nonzero $a \in R$ there are at most a finite number of $i \in \mathcal{I}$ such that a is not a unit of R_i, and (iii) each R_i is a Krull domain. Then R is a Krull domain.

Proof. A Krull domain is an integral domain R that satisfies the following properties (Definition 13):

1. R is completely integrally closed.
2. The ACC holds for the collection of divisorial integral ideals of R.

The first property for R holds since R is the intersection of completely integrally closed domains (Proposition 5). The second property for R holds by the previous proposition. $\square$

10 Local characterizations of Krull domains

We define a Krull domain (Definition 13) as an integral domain R that satisfies the following properties:

1. R is completely integrally closed.
2. The ACC holds for the collection of divisorial integral ideals of R.

We established a second characterization of a Krull domain as an integral domain where each integral divisor has a unique factorization into prime divisors (Theorem 45). In this section we give two other characterizations. We apply the last of these to study of localizations of a Krull domain.

Theorem 86. *Let R be an integral domain with fraction field K, and assume that R is not all of K. Then R is a Krull domain if and only if (i)*

$$R = \bigcap_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$$

where the intersection is indexed by the set of minimal prime ideals of R, (ii) for each minimal prime $\mathfrak{p}$ the localization $R_{\mathfrak{p}}$ is a DVR, and (iii) each nonzero $a \in R$ is contained in only a finite number of minimal prime ideals of R.

Proof. Suppose R is a Krull domain. Then the first condition holds by Corollary 70. The second condition holds by Theorem 68. The third condition holds by Corollary 61.

Now suppose each of the three conditions holds. Observe that each $R_\mathfrak{p}$ is a Krull domain since DVRs are Krull domains (Proposition 42). Observe that if $a \notin \mathfrak{p}$ then a is a unit in $R_\mathfrak{p}$. Thus R is a Krull domain by Corollary 85. $\square$

We can extend this characterization as follows:

Theorem 87. *Let R be an integral domain with field of fractions K, and assume that R is not all of K. Then R is a Krull domain if and only if there is a nonempty family $(R_i)_{i\in\mathcal{I}}$ of DVRs contained in K such that (i)*

$$R = \bigcap_{i\in\mathcal{I}} R_i,$$

and (ii) each nonzero $a \in R$ is a non-unit of R_i for only a finite number of $i \in \mathcal{I}$.

Proof. Suppose both of these conditions holds. Observe that each R_i is a Krull domain since DVRs are Krull domains (Proposition 42). Thus R is a Krull domain by Corollary 85.

Conversely, if R is a Krull domain, let $\mathcal{I}$ be the set of all minimal prime ideals of R. For each $i = \mathfrak{p} \in \mathcal{I}$ let R_i be the localization $R_\mathfrak{p}$. By the preceding theorem, this gives a family of DVRs whose intersection is R. Moreover, each nonzero $a \in R$ is contained in only a finite number of minimal prime ideals of R (Corollary 61), so each such a is a non-unit of $R_\mathfrak{p}$ for only a finite number of $\mathfrak{p} \in \mathcal{I}$. $\square$

Remark. If the above, if we define the intersection of an empty family to be K then we can include the case where $R = K$ is a field. We adopt this convention in the following proposition:

Proposition 88. *Let R be a Krull domain with field of fractions K. Assume that R is not a field. Suppose $(R_i)_{i\in\mathcal{I}}$ is a nonempty family of DVRs contained in K such that (i)*

$$R = \bigcap_{i\in\mathcal{I}} R_i,$$

and (ii) each nonzero $a \in R$ is a non-unit of R_i for only a finite number of $i \in \mathcal{I}$.

If S is a multiplicative system of R, then let $\mathcal{I}_S$ be the collection of $i \in \mathcal{I}$ such that S is contained in the units of R_i. Then

$$S^{-1}R = \bigcap_{i\in\mathcal{I}_S} R_i.$$

Proof. Observe that if $i \in \mathcal{I}_S$ then $S^{-1}R \subseteq R_i$. So $S^{-1}R \subseteq \bigcap R_i$ where the intersection is over $i \in \mathcal{I}_S$.

So suppose x is a nonzero element in this intersection: $x \in R_i$ for all $i \in \mathcal{I}_S$. For each $i \in \mathcal{I}$ let v_i be the discrete valuation associated with R_i. So $v_i(x) \geq 0$ for all $i \in \mathcal{I}_S$. Let $\mathcal{E}_x$ be the set of indices in $\mathcal{I}$ such that $v_i(x) < 0$. Observe that if $i \in \mathcal{E}_x$ then $i \notin \mathcal{I}_S$, and so S must contain an element s_i such that $v_i(s_i) > 0$. Note that $\mathcal{E}_x$ must be a finite set. To verify this write x as a/b with $a, b \in R$, and note that by assumption a and b, and hence x, are units of R_i for all but a finite number of $i \in \mathcal{I}$.

Let s be the product of s_i for $i \in \mathcal{E}_x$ (where $s = 1$ if $\mathcal{E}_x$ is empty). Observe that there is a positive k such that $v_i(s^k x) \geq 0$ for all $i \in \mathcal{E}_x$. Observe that, in fact, $v_i(s^k x) \geq 0$ for all $i \in \mathcal{I}$. Thus $a = s^k x \in R$. So $x = a/s^k \in S^{-1}R$. □

Corollary 89. *Let R be a Krull domain and let S be a multiplicative system of R. Then $S^{-1}R$ is a Krull domain.*

Proof. Apply Theorem 87 to the family $(R_i)_{i \in \mathcal{I}_S}$ of the above proposition. (If this is an empty family, then $R = K$. So R is a field in this case, which we consider to be a Krull domain). □

11 Sequel

There are several other interesting results concerning Krull domains that are worth exploring at this point. I plan to update this essay with additional results at a later date, or to write a separate sequel.