The Krull–Akizuki Theorem

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December 2019

Suppose that R is a Noetherian integral domain with fraction field K. Let $\widetilde{R}$ be the integral closure of R in K. This ring $\widetilde{R}$, called the *normalization* of R, plays an important role in commutative algebra and has applications to number theory and algebraic geometry.

For many of the Noetherian integral domains R that occur in practice the ring $\widetilde{R}$ will be a finitely generated R-module. This will insure that $\widetilde{R}$ is itself Noetherian (see Proposition 14). A big shock is that $\widetilde{R}$ is not always finitely generated over R, and is not even Noetherian in general. What makes this result intriguing is that this does not happen in the standard situations, and one has to work hard to find counterexamples. Hideyuki Matsumura (in *Commutative ring theory*) mentions that Yasuo Akizuki searched a full year for a counterexample. Miles Reid (in *Undergraduate Commutative Algebra*) gives an accessible account of counterexamples including one of Akizuki's counterexamples where R is a local Noetherian integral domain of dimension one (every nonzero prime ideal is maximal).

The Krull-Akizuki theorem saves us in the situation where R is a Noetherian integral domain *and* where R has another nice property: every nonzero prime ideal of R is maximal. It does not assert that $\widetilde{R}$ will be a finitely generated R-module, but it will guarantee that $\widetilde{R}$ is at least Noetherian. This will make $\widetilde{R}$ into a well-behaved ring: a Dedekind domain.[1]

The Krull-Akizuki theorem asserts much more: Suppose R is a Noetherian integral domain with fraction field K, and suppose L is a finite extension of K. Assume, as mentioned above, that every nonzero prime ideal of R is maximal. The Krull–Akizuki theorem asserts that any ring R' with $R \subseteq R' \subseteq L$ is also Noetherian and also has the property that every nonzero prime ideal of R' is maximal. It is not necessarily true that R' will be a finitely generated R-module, but the theorem asserts that R'/I will be a finitely generated R-module for any nonzero ideal I.

Yasuo Akizuki is known for another theorem: every Artinian ring is a Noetherian ring. This theorem is easier to prove than the Krull-Akizuki, but it is still a significant result. This essay gives proofs of both these results bearing Akizuki's

[1] According to David Eisenbud (*Commutative Algebra with a View Toward Algebraic Geometry*, page 263 and page 294) Mori and Nagata showed that $\widetilde{R}$ is guaranteed to be Noetherian even in dimension 2, but their conclusion is weaker than the conclusion of the Krull-Akizuki theorem for the dimension 1 case. In dimension 3 it is possible that the normalization of a Noetherian integral domain fails to be Noetherian. A reference to these results is M. Nagata (*Local Rings*).

name. (Akizuki contributions to both results appear in papers published in 1935). The aim here is to give a friendly approach of these results. A friendly approach is not necessarily the shortest or most economical approach. Rather it is an approach that is designed to be easy for the reader to read, digest, and appreciate. The proofs should be, as much as possible, transparent, natural, and avoid unnecessary cleverness. The exposition should be divided up, if possible, into easily digestible steps, and the ideas involved should be as straightforward and natural as possible. For example, the proofs here lean heavily on the notion of lengths of modules since this is an intuitive and natural notion that has applications far beyond the theorems presented here.

The proofs of these main results rely on preliminary propositions and lemmas that are sometimes straightforward to prove. My philosophy is that, at this level of mathematics, such straightforward proofs are best worked out by the reader. So, whenever a proof is not given or is sketchy, this signals to the reader that they should work out the proof and that the details are reasonably straightforward.

1 General background

I assume the reader is comfortable with the basics of a standard year-long sequence in abstract algebra including the basics of commutative rings and their modules, some linear algebra over an abstract field, and some basic field theory. The reader should be familiar with the notion of finitely generated modules and ideals, and should be used to viewing an ideal or a ring extension as a module. Recall that a vector space is just a module over a field, and is finitely generated if and only if it is finite dimensional.

Assumed background includes the following facts. There is a natural correspondence between submodules of a quotient module M/N and submodules of M containing N. Similarly, there is a natural correspondence between ideals of a quotient ring R/I and ideals of R containing I. This correspondence of ideals restricts to a correspondence for prime ideals and for maximal ideals. Also, a module homomorphism $M \to N$ with kernel K induces a natural injective homomorphism $M/K \to N$.

In addition to this standard background, I assume as background the more specialized material in my essay *Chains and Lengths of Modules* including the background assumed there. In particular, the reader is assumed to be familiar with the ascending chain conditions (ACC) and the descending chain condition (DCC). Modules satisfying the first condition are called *Noetherian* and modules satisfying the second condition are called *Artinian*. Modules satisfying both are said to have *finite length*. Some basic properties of length from this earlier essay are reviewed in the next section. Note: the ACC is equivalent to the assumption that every nonempty collection of submodules has a maximal element, and the DCC is equivalent to the assumption that every nonempty collection of submodules has a minimal element.

Let R be an integral domain with fraction field K. At some point (starting with Lemma 9) we will be interested in R-submodules of K. If I is an R-submodule of K and if $x \in K$ then we define xI to be the set $\{xy \mid y \in I\}$. Observe that xI is a also an R-submodule of K. A *fractional ideal* I is a nonzero R-submodule of K

such that dI is an ideal for some nonzero $d \in R$. For example, nonzero finitely generated R-submodules of K are fractional ideals.

The reader should be familiar with prime ideals of a commutative ring together with the fact that $\mathfrak{p}$ is a prime ideal of a commutative ring R if and only if $R/\mathfrak{p}$ is an integral domain. Similarly the reader should be familiar with maximal ideals of a commutative ring together with the fact that $\mathfrak{m}$ is a maximal ideal of a commutative ring R if and only if $R/\mathfrak{m}$ is a field.

The reader should be familiar with the products of ideals in a commutative ring. If I and J are ideals of R then IJ is the set of finite sums of elements of the form ab with $a \in I$ and $b \in J$. In other words, IJ is the ideal generated by such products ab. If $I_1 I_2 \cdots I_k$ is a product of ideals and if $I_1 I_2 \cdots I_k \subseteq \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal then $I_i \subseteq \mathfrak{p}$ for some I_i.

If I is an ideal of a commutative ring R and if R' is a commutative ring containing R as a subring, then IR' is the collection of finite sums with terms of the form ab with $a \in I$ and $b \in R'$. It is an ideal of R'; it is in fact the ideal of R' generated by the elements of I.

If R is a subring of a ring A, then an element $\alpha \in A$ is said to be *integral* over R if it is the root of a monic polynomial in $R[X]$. In this case the ring $R[\alpha]$ is a finitely generated R-module, and $1, \alpha, \ldots, \alpha^{d-1}$ are generators where d is the degree of such a monic polynomial. More generally, given $\alpha_1, \ldots, \alpha_k$ in such an A, the ring they generate $R[\alpha_1, \ldots, \alpha_k]$ is a finitely generated R-module. The following converse is optional but is used in one version of one of our proofs: *If R is a commutative ring and if A is a commutative ring extending R that is finitely generated as an R-module, then every element of A is integral over R.*

We assume some basic facts about field theory. For example, if K is a field and if α algebraic over K (in some extension of K), then α has a unique monic minimal polynomial $f \in K[X]$ which is irreducible and has root α. In this case the ring $K[\alpha]$ is a field, and its degree $[K[\alpha] : K]$ is equal to the degree d of the minimal polynomial. Furthermore, $K[\alpha]$ is a K-vector space with basis $1, \alpha, \alpha^2, \ldots, \alpha^{d-1}$. If L is a finite degree field extension of K and if E is a finite degree field extension of L then E is a finite degree field extension of K and

$$[E : K] = [E : L][L : K].$$

2 Review of modules

In this section we review (without proof) some of the basic results concerning modules, especially for modules of finite length. See my essay *Chains and Lengths of Modules* for more information and some details about the proofs. Throughout this section R is a commutative ring.

Let M be an R-module. We say that M is *simple* if M is a nonzero module and if its only R-submodules are 0 and M itself. We say that M has *finite length* if it has a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

where M_i/M_{i-1} is a simple module for each $i \in \{1, \ldots, n\}$. Here n is called the *length* of the composition series.

Lemma 1. *Let M be a R-module. Then M has finite length over R if and only if M is an R-module that is both Noetherian and Artinian. If M has finite length then all composition series for M have the same length.*

Definition 1. Let M be a R-module. Then the length of M is the length of a composition series of M. We write $\text{length}(M)$ or $\text{len}(M)$ for this length.

Lemma 2. *Let M be an R-module with submodule N. Then M is Noetherian if and only if N and M/N are Noetherian. Similarly, M is Artinian if and only if N and M/N are Artinian. Thus M is has finite length if and only if N and M/N have finite length.*

Lemma 3. *Let M be an R-module, and let*

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k$$

be a finite chain of submodules of M. Then M_k/M_0 has finite length if and only if M_i/M_{i-1} has finite length for each $i \in \{1, \ldots, k\}$. In this case,

$$\text{len}(M_k/M_0) = \text{len}(M_k/M_{k-1}) + \ldots + \text{len}(M_2/M_1) + \text{len}(M_1/M_0).$$

Lemma 4. *Suppose M is an R-module and suppose I is an ideal of R that annihilates M in the sense that $aM = 0$ for each $a \in I$. Then M is naturally an R/I-module. In this case an Abelian subgroup N of M is an R-submodule of M if and only if N is an R/I-submodule of M.*

The module M is Noetherian as an R-module if and only if it is Noetherian as an R/I-module. Likewise, M is Artinian as an R-module if and only if it is Artinian as an R/I-module. Similarly, M has finite length as an R-module if and only if it has finite length as an R/I-module. In this case the lengths are the same (as an R-module versus as an R/I-module).

Lemma 5. *Let M be an F-vector space where F is a field. Then M has finite length if and only if M is a finite dimensional F-vector space. In this case the length is just the dimension. In fact, for such vector spaces M the following are equivalent:*

1. *M has finite length.*
2. *M has finite dimension.*
3. *M is Noetherian.*
4. *M is Artinian.*

Lemma 6. *A module M fails to have finite length if and only if for all $n \in \mathbb{N}$ there is a strict finite chain of submodules of M*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n.$$

Moreover, if M has finite length then the length of such a strict chain is bounded by the length of M.

Lemma 7. *If M is a Noetherian R-module then M is a finitely generated R-module. In particular, if M is an R-module of finite length then M is a finitely generated R-module.*

Lemma 8. *Let M be a module over a commutative ring R. Then the following are equivalent.*

1. *M is simple.*
2. *M has has finite length over R with $\text{length}(M) = 1$.*
3. *M is isomorphic as an R-module to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R.*

3 Additional lemmas concerning modules

Now we consider a few more lemmas about modules that we will need in this document.

Lemma 9. *Suppose R is an integral domain and suppose I is an R-submodule of the fraction field K of R. Suppose also that J is an R-submodule of I and that $x \in K^\times$. Then xI is an R-submodule of K, and xJ is an R-submodule of xI. Furthermore*

$$I/J \;\cong\; xI/xJ.$$

Proof. It is straightforward that xI is an R-submodule of K and xJ is an R-submodule of xI.

Define a map $I \to xI$ by the rule $y \mapsto xy$. Observe that this map is a module homomorphism. This map is surjective, and since $x \neq 0$ in K it must be injective as well. So this gives an isomorphism.

Now compose the isomorphism $I \to xI$ with the quotient map $xI \to xI/xJ$. This is a surjection $I \to xI/xJ$ mapping y to $[xy]$. The kernel is the set of $y \in I$ such that $xy \in xJ$. So the kernel is just J. Thus I/J is isomorphic to xI/xJ. □

Lemma 10. *Suppose R' is a commutative ring with subring R and suppose M is an R'-module. Then M is also an R-module. Furthermore, if M is an R-module of finite length then M is an R'-module of finite length.*

Proof. This is straightforward given Lemma 6. □

4 Noetherian and Artinian rings

We now consider a few basic results about Noetherian and Artinian rings.

Definition 2. A commutative ring R is a *Noetherian ring* if R is a Noetherian R-module. A commutative ring R is an *Artinian ring* if R is a Artinian R-module.

In other words, a ring R is Noetherian if and only if the collection of ideals of R satisfy the ACC. Similarly, the ring R is Artinian if and only if the collection of ideals of R satisfy the DCC.

Definition 3. An integral domain that is Noetherian will be called a *Noetherian domain*. An integral domain that is Artinian will be called a *Artinian domain*.

The following result is straightforward using the correspondence between ideals of a quotient ring R/I and ideals of R containing I.

Lemma 11. *Let I be an ideal in a commutative ring R. If R is a Noetherian ring then so is R/I. If R is an Artinian ring then so is R/I.*

Proposition 12. *A commutative ring R is Noetherian if and only if every ideal of R is finitely generated.*

Proof. Suppose R is Noetherian and let I be an ideal of R. Then I is finitely generated as an R-module by Lemma 7.

Now suppose every ideal of R is finitely generated. Given an ascending chain of ideals in R, the union is a finitely generated ideal of R, so the chain stablizes. □

Proposition 13. *Suppose M is a module over a Noetherian ring R. Then M is Noetherian if and only if M is finitely generated as an R-module.*

Proof. If M is Noetherian then M is finitely generated (Lemma 7).

Conversely suppose that M is finitely generated. We will show by induction on the minimal number of generators k of M that M is Noetherian. (We consider that $M = 0$ has zero generators, so this case is clear). Suppose that M has generating set S of minimal size $k + 1$, and let $s \in S$. Then M/sR is Noetherian by induction. Consider the R-module homomorphism $R \to sR$ given by $x \mapsto sx$, and let I be the kernel. So sR is isomorphic to R/I as an R-module. Since R is a Noetherian ring, R is a Noetherian R-module. Thus R/I and hence sR are Noetherian modules (Lemma 2). Since sR and M/sR are Noetherian, M is as well (Lemma 2). □

Proposition 14. *Suppose that R is a Noetherian ring and suppose that R' is a commutative ring extending R that is a finitely generated R-module. Then R' is a Noetherian ring.*

Proof. Since R' is a finitely generated R-module, R' is a Noetherian R-module by the previous proposition. Every ideal I of R' is a R-submodule of R', so is a Noetherian R-module (Lemma 2). Thus every ideal I of R' is finitely generated as an R-module by the previous proposition. In particular, every ideal I of R' is finitely generated as an R'-module. So R' is a Noetherian ring (Proposition 12). □

In what follows we are particular interested in Noetherian rings that are "zero dimensional" in the sense that every prime ideal is maximal. These are important because they turn out to be the Artinian rings.

In the following we allow empty products. The empty product of ideals is defined to be the whole ring R.

Lemma 15. *Let I be an ideal of a Noetherian ring R. Then there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, where $k \geq 1$, such that*

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq I.$$

Proof. Suppose there are ideals where this fails. Using the ACC, there is a maximal such ideal I. Note that I is a proper ideal that is not a prime ideal, so there are $x, y \in R$ such that $xy \in I$ but x, y are not in I. We can assume the result for $I + xR$ and $I + yR$. Finally, observe that $(I + xR)(I + yR) \subseteq I$. □

Corollary 16. *The zero ideal of a Noetherian ring is the product of prime ideals.*

Corollary 17. *Let R be a Noetherian ring such that every prime ideal is maximal. Then R has only a finite number of prime ideals.*

Proof. Write $\mathfrak{p}_1 \cdots \mathfrak{p}_k = 0$ for some finite sequence of prime ideals $\mathfrak{p}_i$. If $\mathfrak{p}$ is a prime ideal then $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{p}$. So $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. Hence $\mathfrak{p} = \mathfrak{p}_i$ since $\mathfrak{p}_i$ is maximal. □

Proposition 18. *Every Artinian domain R is a field.*

Proof. Let $x \in R$ be a nonzero element. Consider the descending chain

$$xR \supseteq x^2R \supseteq x^3R \supseteq \cdots .$$

This chain must stabilize. This fact can be used to find an inverse for x in R. □

Corollary 19. *Any prime ideal $\mathfrak{p}$ of an Artinian ring R is maximal.*

Proof. The ring $R/\mathfrak{p}$ is an Artinian domain, hence is a field. □

Proposition 20. *Let R be an Artinian ring and let I be an ideal of R that is minimal with respect to the property of being a product of prime ideals. Then I is contained in all prime ideals of R. Furthermore, $I^2 = I$.*

Proof. Let $\mathfrak{p}$ be a prime ideal. Observe that $\mathfrak{p}I$ is the product of prime ideals. Since $\mathfrak{p}I \subseteq I$ this implies $\mathfrak{p}I = I$ by the minimality of I. Thus $I \subseteq \mathfrak{p}$.

Similarly I^2 is the product of prime ideals and $I^2 \subseteq I$. So $I^2 = I$ by the minimality of I. □

Proposition 21. *Any Artinian ring R has a finite number of prime ideals.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. If I is as in the previous proposition then $I \subseteq \mathfrak{p}$. Write $I = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ where each $\mathfrak{p}_i$ is a prime ideal. So $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{p}$. Thus $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. Since every prime ideal of R is maximal, we have $\mathfrak{p} = \mathfrak{p}_i$ (Corollary 19). □

Theorem 22. *Let R be a Noetherian ring such that every prime ideal is maximal. Then R is Artinian.*

Proof. We argue via the following steps:

- The zero ideal of R is the product of prime ideals: $0 = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ (Corollary 16).
- Given such a factorization $\mathfrak{p}_1 \cdots \mathfrak{p}_k = 0$, let $I_i = \mathfrak{p}_1 \cdots \mathfrak{p}_i$ and let $I_0 = R$. In other words, $I_i = I_{i-1}\mathfrak{p}_i$ if $i > 0$. We get a finite chain

$$0 = I_k \subseteq I_{k-1} \subseteq \cdots \subseteq I_1 \subseteq I_0 = R.$$

- Each I_i is a Noetherian R-module since it is a submodule of R, and R is assumed to be Noetherian. (Lemma 2)
- Each quotient I_i/I_{i+1} is the quotient of a Noetherian R-module, so must be a Noetherian R-module. (Lemma 2)
- Since $\mathfrak{p}_{i+1}$ annihilates $I_i/I_{i+1} = I_i/\mathfrak{p}_{i+1}I_i$, the R-module I_i/I_{i+1} is an $R/\mathfrak{p}_{i+1}$-module. (Lemma 4)
- The quotient I_i/I_{i+1} is a Noetherian $R/\mathfrak{p}_{i+1}$-module since it is a Noetherian R-module. (Lemma 4)
- Every prime ideal of R is maximal, so $R/\mathfrak{p}_{i+1}$ is a field. Thus I_i/I_{i+1} is a vector space over the field $R/\mathfrak{p}_{i+1}$.
- Each quotient I_i/I_{i+1} is a finite dimensional vector space over the field $R/\mathfrak{p}_{i+1}$ since it is a Noetherian $R/\mathfrak{p}_{i+1}$-module. In particular, I_i/I_{i+1} has finite length as an $R/\mathfrak{p}_{i+1}$ module. (Lemma 5)
- Each quotient I_i/I_{i+1} has finite length as an R-module. (Lemma 4)
- $R \cong I_0/I_k$ has finite length as an R-module (Lemma 3).
- Thus R is both Artinian and Noetherian as an R-module. (Lemma 1)

□

5 Finite extensions of a ring

Our main goal, the Krull-Akizuki theorem, applies to a tricky situation where we have an extension of a ring R that is not necessarily a finitely generated R-module. When we do have an extension ring, A say, which is a finitely generated R-module, matters are relatively well-behaved even for quite general rings R. In this section we consider this tame situation before we consider the Krull-Akizuki situation.

We start with special cases where one of the two rings is a field.

Lemma 23. *Suppose K is a field and A is a finite dimensional extension of K that is an integral domain. Then A is a field.*

Proof. Since A has finite length as a K-module (Lemma 5), it has finite length as an A-module (Lemma 10). So A is an Artinian domain (Lemma 1), which means that A must be a field (Proposition 18).

Note: There is a second argument that uses integrality of A over K and Lemma 25 below. □

There is a converse to the above lemma:

Lemma 24. *Suppose R is a subring of a field L. If L is finitely generate over R then R is also a field.*

First proof. Let K be the fraction field of R. Our goal is to show that $K = R$. Our proof will be by strong induction on $k = [L : K]$. If $k = 1$ then K is L and is finitely generated as an R-module. This implies that there is an element $d \in R$ not zero such that $dK \subseteq R$. However $dK = K$. Thus $K = R$.

Now suppose $k > 1$, and let $x \in L \setminus K$. Let $K' = K[x]$. By basic field theory, the ring K' is a field and, if $n = [K' : K]$, then $n > 1$ and $[L : K'] = k/n < k$. Let $f \in K[X]$ be the monic minimal polynomial of x. There is a $r \in R$ nonzero so that $rf \in R[X]$. It follows that $y = rx$ has monic minimal polynomial in $R[X]$ of degree n and $K' = K[y]$.

Observe that the ring $R[y]$ is an R-module with generating set $1, y, \ldots, y^{n-1}$. Since the fraction field of $R[y]$ is K' and since $[L : K'] < k$ we can assume that $R[y]$ is a field since L is finitely generated as an $R[y]$-module. So $K \subseteq R[y]$. Since K' is $K[y]$, every element of K' can be written uniquely as a K-linear combination of $1, y, \ldots, y^{n-1}$.

Let $c \in K$. Then $c = c_0 + c_1 y + \ldots + c_{n-1} y^{n-1}$ for some $c_i \in R$ since $c \in R[y]$. By uniqueness of representative, $c = c_0$. Thus $c \in R$. We conclude that $R = K$. □

Second proof. The second proof uses less field theory, but more about integral elements. It uses the following fact: *if R is a subring of a commutative ring A and if A is a finitely generated R-module, then A is an integral extension of R.* We won't prove this fact here, but just observe that it enough to prove the result by the following lemma. □

Lemma 25. *Suppose A is an integral domain that is an integral extension of R. Then A is a field if and only if R is a field.*

Proof. Suppose that A is a field. Let $r \in R$ be a nonzero element and let $a \in A$ be its multiplicative inverse. Since A is an integral extension,

$$a^n + r_{n-1}a^{n-1} + \ldots + r_1 a + r_0 = 0$$

for some $n \geq 1$ and $r_1, \ldots, r_{n-1} \in R$. So

$$a = -r^{n-1}(r_{n-1}a^{n-1} + \ldots + r_1 a + r_0) = -(r_{n-1} + \ldots + r^{n-2}r_1 + r^{n-1}r_0).$$

So $a \in R$. This establishes that R is a field.

Suppose R is a field. Let $a \in A$ be a nonzero element. Since A is an integral extension,

$$a^n + r_{n-1}a^{n-1} + \ldots + r_1 a + r_0 = 0$$

for some $n \geq 1$ and $r_1, \ldots, r_{n-1} \in R$. By cancelling by a power of a if necessary, we can assume $r_0 \neq 0$. Thus

$$a(a^{n-1} + r_{n-1}a^{n-2} + \ldots + r_1)(-r_0)^{-1} = 1.$$

So a has an inverse in A. This establishes that A is a field. □

We will need the following lemma in a few places:

Lemma 26. *Suppose A is a commutative ring with subring R. Suppose $\mathfrak{m}_0$ is a maximal ideal of R such that $A/\mathfrak{m}_0 A$ is a finitely generated R-module. Then there are only a finite number of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap R = \mathfrak{m}_0$. Each such prime ideal is a maximal ideal.*

Proof. Observe that a proper ideal I of A has the property that $I \cap R = \mathfrak{m}_0$ if and only if I contains $\mathfrak{m}_0 A$ (one implication uses the fact that $\mathfrak{m}_0$ is maximal). We are thus interested in prime ideals of A containing $\mathfrak{m}_0 A$. These correspond to prime ideals of $A/\mathfrak{m}_0 A$.

Since $A/\mathfrak{m}_0 A$ is a finitely generated R-module, it is a finitely generated $R/\mathfrak{m}_0$-module (Lemma 4). Now $R/\mathfrak{m}_0$ is a field, and $A/\mathfrak{m}_0 A$ is a vector space over this field. In fact, $A/\mathfrak{m}_0 A$ is a finite dimensional vector space over this field. So $A/\mathfrak{m}_0 A$ has finite length over this field (Lemma 5). Thus $A/\mathfrak{m}_0 A$ has finite length over R (Lemma 4), so has finite length over A (Lemma 10), and thus has finite length over $A/\mathfrak{m}_0 A$ (Lemma 4). Hence $A/\mathfrak{m}_0 A$ is an Artinian ring (Lemma 1). So $A/\mathfrak{m}_0 A$ has only a finite number of prime ideals (Proposition 21), and every such prime ideal is maximal (Corollary 19). □

Proposition 27. *Suppose A is a commutative ring with subring R. Then the function $\mathfrak{p} \mapsto \mathfrak{p} \cap R$ maps the set of prime ideals of A to the set of prime ideals of R.*

If A is finitely generated as an R-module and if $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}$ is maximal if and only if $\mathfrak{p} \cap R$ is a maximal ideal of R. For every maximal ideal $\mathfrak{m}_0$ of R there are only a finite number of maximal ideals $\mathfrak{m}$ of A such that $\mathfrak{m} \cap R = \mathfrak{m}_0$.

Similarly, if A is an integral extension of R and if $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}$ is maximal if and only if $\mathfrak{p} \cap R$ is a maximal ideal of R.

Proof. If $\mathfrak{p}$ is a prime ideal of A then $\mathfrak{p} \cap R$ is a prime ideal of R; this is straightforward to verify.

Let A be finitely generated as an R-module, let $\mathfrak{p}$ be a prime ideal of A, and let $\mathfrak{p}_0 = \mathfrak{p} \cap R$. Then $\mathfrak{p}_0$ is the kernel of the composition $R \to A \to A/\mathfrak{p}$ (the inclusion followed by the quotient map). So we can identify $R/\mathfrak{p}_0$ with an integral domain contained in the integral domain $A/\mathfrak{p}$. Observe that $A/\mathfrak{p}$ is finitely generated as an $R/\mathfrak{p}_0$-module. By Lemmas 23 and 24 we have $A/\mathfrak{p}$ is a field if and only if $R/\mathfrak{p}_0$ is a field. Thus $\mathfrak{p}$ is a maximal ideal if and only if $\mathfrak{p}_0$ is a maximal ideal. This argument can be adapted to the situation of A integral over R using Lemma 25 instead of Lemmas 23 and 24.

Now let $\mathfrak{m}_0$ be a maximal ideal of R. Since A is finitely generated as an R-module, $A/\mathfrak{m}_0$ is finitely generated as an R-module. So by the previous lemma, there are only finitely many maximal ideals $\mathfrak{m}$ of A with $\mathfrak{m} \cap R = \mathfrak{m}_0$. □

Proposition 28. *Suppose A is an integral domain with subring R. Suppose every element of A is algebraic over the fraction field K of R. Then the only ideal I of A such that $I \cap R = 0$ is the zero ideal 0 of A.*

Proof. Let I be a nonzero ideal of A and let $a \in I$ be a nonzero element. Since a is algebraic over K, there is a nonzero minimal polynomial $f \in K[X]$ with $f(a) = 0$. Since $a \neq 0$, the constant term of f is a nonzero element of K. Clear denominators by multiplying f by some nonzero $d \in R$. This results in a polynomial $g \in R[X]$

with nonzero constant term $c \in R$ such that $g(a) = 0$. This yields an expression for c that shows that $c \in I$. So $I \cap R \neq 0$. □

Proposition 29. *Suppose A is a commutative ring and R is a subring of A such that A is a finitely generated R-module. If M is a module of finite length over A then M is a module of finite length over R.*

Proof. We start with a simple module $M = A/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of A. Let $\mathfrak{m}_0 = \mathfrak{m} \cap R$. By Proposition 27, $\mathfrak{m}_0$ is a maximal ideal of R, so $F = R/\mathfrak{m}_0$ is a field. Observe that M is an F-vector space (Lemma 4). Since A is finitely generated over R, this vector space M has finite dimension. So M has finite length as an F-module (Lemma 5), and as an R-module (Lemma 4).

Since all simple A-modules are isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ (Lemma 8), all simple A-modules have finite length over R.

Now let M be any A-module of finite length. Consider any composition series for M, and consider this as a chain of R-modules. By using Lemma 3 we can conclude that M has finite length as an A-module. □

6 The Krull-Akizuki theorem

At this point, we have built up enough basic commutative algebra to give the Krull-Akizuki a fairly natural, straightforward proof. The key to the Krull-Akizuki theorem is the following two propositions:

Proposition 30. *Suppose R is a Noetherian domain such that every nonzero prime ideal is maximal, and suppose $a \in R$ is a nonzero element. If I is a fractional ideal of R then R/aR and I/aI are R-modules of finite length and*

$$\text{length}(I/aI) = \text{length}(R/aR).$$

Proof. If a is a unit, the result is straightforward, so we assume a is not a unit. We consider first the case of I a nonzero ideal. Using the correspondence between prime ideals of R/aI and prime ideals of R containing aI we see that R/aI is a Noetherian ring such that every prime ideal is maximal. Thus R/aI is also an Artinian ring (Theorem 22). This means that R/aI is a Noetherian and Artinian module over R (Lemma 4), so R/aI has finite length (Lemma 1). Since $aI \subseteq aR \subseteq R$, the quotients R/aR and aR/aI have finite length and

$$\text{len}(R/aI) = \text{len}(R/aR) + \text{len}(aR/aI) = \text{len}(R/aR) + \text{len}(R/I)$$

(see Lemma 3 and Lemma 9). Since $aI \subseteq I \subseteq R$, the quotients R/I and I/aI have finite length and

$$\text{len}(R/aI) = \text{len}(R/I) + \text{len}(I/aI)$$

(see Lemma 3). These two equations combine to yield the following equation:

$$\text{len}(R/aR) = \text{len}(I/aI).$$

This establishes the result for nonzero ideals I.

For a fractional ideal I we have that bI is an ideal for some nonzero $b \in R$. Since bI/abI is isomorphic to I/aI (Lemma 9), I/aI must have finite length and

$$\text{len}(I/aI) = \text{len}(bI/abI) = \text{len}(R/aR).$$

□

We now extend this result to any R-submodule of the fraction field K.

Proposition 31. *Suppose R is a Noetherian domain with fraction field K. Suppose every nonzero prime ideal is maximal, and suppose $a \in R$ is a nonzero element. If I is an R-submodule of K then I/aI and R/aR are R-modules of finite length and*

$$\text{length}(I/aI) \leq \text{length}(R/aR).$$

Proof. We know that R/aR has finite length by the previous proposition. Suppose that the R-module I/aI fails to have finite length or has length strictly greater than k where $k = \text{length}(R/aR)$. Then there is a strict chain of length $k+1$:

$$aI \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{k+1} \subseteq I$$

(Lemma 6). Replace I_1 with $aI + b_1R$ for some $b_1 \in I_1 \smallsetminus aI$. Replace I_2 with

$$I_1 + b_2R = aI + b_1R + b_2R$$

for some $b_2 \in I_2 \smallsetminus I_1$ and so on. After replacement

$$I_{k+1} = aI + b_1R + b_2R + \ldots + b_{k+1}R = aI + J$$

where $J = b_1R + b_2R + \ldots + b_{k+1}R$. Observe that J is a fractional ideal (since it is finitely generated), and that $J \subseteq I$. So J/aJ has finite length by the previous proposition and

$$aJ \subsetneq aJ + b_1R \subsetneq aJ + b_1R + b_2R \subsetneq \cdots \subsetneq J.$$

This implies that J/aJ has length at least $k+1$ (Lemma 6), contradicting the previous proposition. □

Corollary 32. *Suppose R is a Noetherian domain with fraction field K. Suppose that every nonzero prime ideal of R is maximal, and suppose $a \in R$ is a nonzero element. If R' is a subring of K containing R then R'/aR' has finite length as an R-module.*

Proposition 33 (Weak Krull-Akizuki)**.** *Suppose R is a Noetherian domain such that every nonzero prime ideal is maximal. If R' is a subring of the fraction field K of R, and if R' contains R, then R' is also a Noetherian domain such that every nonzero prime ideal is maximal. Furthermore, if I is a nonzero ideal of R' then the quotient R'/I is an R-module of finite length.*

Proof. Let I be a nonzero ideal of R' and let $a \in I$ be nonzero. By the previous corollary R'/aR' is an R-module of finite length. Since $aR' \subseteq I \subseteq R'$ we have that R'/I and I/aR' are both R-modules of finite length (Lemma 3).

Since I/aR' has finite length as an R-module, then it has finite length as an R'-module (Lemma 10). This implies that I/aR' is a finitely generated R'-module (Lemma 7). If $b_1, \ldots, b_k \in I$ are such that I/aR' is generated by $[b_1], \ldots, [b_k]$, then $I = aR' + b_1R' + \ldots + b_kR'$, and so I is finitely generated.

Since every nonzero ideal I of R' is finitely generated, the ring R' is Noetherian (Proposition 12).

Let $\mathfrak{p}$ be a nonzero prime ideal of R'. Then, as above, $R'/\mathfrak{p}$ has finite length as an R-module. So it has finite length as an R'-module (Lemma 10), and as an $R'/\mathfrak{p}$ module (Lemma 4). This means that $R'/\mathfrak{p}$ is an Artinian domain (Lemma 3), which in turn means it is a field (Proposition 18). Thus $\mathfrak{p}$ is a maximal ideal of R'. □

To move beyond the weak Krull-Akizuki situation we need to consider rings in finite extension fields L of the fraction field K of R. The following gives us the leverage we need for the more general situation.

Lemma 34. *Suppose R is an integral domain with fraction field K. Let L be a finite extension of K and let R' be a subring of L that contains R. Then there is a ring A such that* (1) *$R \subseteq A \subseteq R'$,* (2) *$A$ is a finitely generated R-module, and* (3) *the integral domains A and R' have the same fraction field inside L.*

Proof. The idea is to define A in terms of integral elements of R'. First we claim that if $\alpha \in R'$ then $a\alpha$ is integral over R for some nonzero $a \in R$. To see this let $f \in K[X]$ be the minimal polynomial of $\alpha \in R'$. By clearing denominators, we can assume $f \in R[X]$. If $a \in R$ is a multiple (in R) of the leading coefficient of f, then it is a straightforward exercise to show that $a\alpha$ is the root of a monic polynomial in $R[X]$, so $a\alpha$ is integral over R.

Let K' be the fraction field of R' in L. We next claim that every $x \in K'$ can be written in the form β/γ where β, γ are in R' and are both integral over R. To see this, first write $x = \alpha_1/\alpha_2$ with $\alpha_1, \alpha_2 \in R'$. Choose $a \in R$ nonzero so that $\beta = a\alpha_1$ and $\gamma = a\alpha_2$ are integral over R. Thus $x = \beta/\gamma$ as desired.

Since K' is a finite extension of K we have $K' = K[x_1, \ldots, x_k]$ for some $x_i \in K'$. Write $x_i = \beta_i/\gamma_i$ where β_i, γ_i are in R' and are integral over R. We define A as follows:

$$A = R[\beta_1, \gamma_1, \ldots, \beta_k, \gamma_k].$$

Observe that $R \subseteq A \subseteq R'$ and that the fraction field of A is K'. Finally, A is a finitely generated R-module since each β_i and each γ_i is integral over R. □

Now we are ready for the main theorem:

Theorem 35 (Krull-Akizuki)**.** *Suppose R is a Noetherian domain such that every nonzero prime ideal is maximal. Suppose that L is a finite extension of the fraction field K of R and suppose that R' is a subring of L containing R. Then R' is also a Noetherian domain where every nonzero prime ideal is maximal.*

Furthermore, if I is a nonzero ideal of R' then the ring R'/I has finite length as an R-module.

Proof. Let A be as constructed in Lemma 34. In particular, (1) $R \subseteq A \subseteq R'$, (2) A is a finitely generated R-module, and (3) the integral domains A and R' have the same field of fractions in L, call it K'. Since A is a finitely generated R-module, it is Noetherian (Proposition 14).

The map $\mathfrak{p} \mapsto \mathfrak{p} \cap R$ sends prime ideals of A to prime ideals of R and any prime ideal of A mapping to a maximal ideal of R is a maximal ideal (Proposition 27). The only prime ideal of A mapping to 0 in R is the zero ideal 0 of A (Proposition 28). This implies that every nonzero prime ideal of A is maximal.

Since A is Noetherian and every nonzero prime ideal of A is maximal, the same must be true of R' by the weak Krull-Akizuki theorem (since A and R' have a common fraction field K').

Let I be a nonzero ideal of R'. Then R'/I has finite length as an A-module by the weak Krull-Akizuki theorem. Since A is finitely generated as an R-module, the R-module R'/I must also have finite length (Proposition 29). □

7 Correspondence of primes

Now we consider the behavior of prime ideals in the Krull-Akizuki situation:

Proposition 36. *Suppose R is a Noetherian domain such that every nonzero prime ideal is maximal, and suppose K is the fraction field of R. Suppose that L is a finite extension of K and that R' is a subring of L containing R. Suppose also that R' is not itself a field. Then the following hold*

1. *The function $\mathfrak{p} \mapsto \mathfrak{p} \cap R$ maps the set of maximal ideals of R' to the set of maximal ideal of R.*
2. *For every maximal ideal $\mathfrak{m}'$ of R', if $\mathfrak{m} = \mathfrak{m}' \cap R$ then there is a natural injective homomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}'$. When we identify $R/\mathfrak{m}$ with a subfield F of $R'/\mathfrak{m}'$, then $R'/\mathfrak{m}'$ is a finite dimensional extension of F.*
3. *For every maximal ideal $\mathfrak{m}$ of R there are only finitely many maximal ideals of R' mapping to $\mathfrak{m}$ under the function $\mathfrak{p} \mapsto \mathfrak{p} \cap R$.*

Proof. It is straightforward to show that the function $\mathfrak{p} \mapsto \mathfrak{p} \cap R$ maps the set of prime ideals of R' to the set of prime ideal of R. Since the only ideal mapping to 0 is the zero ideal (Proposition 28), every maximal ideal of R' must map to a nonzero prime ideal of R. Hence every maximal ideal of R' maps to a maximal ideal of R.

Let $\mathfrak{m}'$ be a maximal ideal of R' and let $\mathfrak{m} = \mathfrak{m}' \cap R$ be the corresponding maximal ideal of R. Observe that $\mathfrak{m}$ is the kernel of the composition $R \to R' \to R'/\mathfrak{m}'$ (the inclusion followed by the quotient map). So we get an injection $R/\mathfrak{m} \to R'/\mathfrak{m}'$. By the Krull-Akizuki theorem $R'/\mathfrak{m}'$ has finite length as an R-module. Thus it has finite dimension as a $R/\mathfrak{m}$-vector space (Lemma 4, Lemma 5). Let F be the image of $R/\mathfrak{m}$ in $R'/\mathfrak{m}'$. Any basis of $R'/\mathfrak{m}'$ as an $R/\mathfrak{m}$-vector space is a basis as an F-vector space. So $R'/\mathfrak{m}'$ is a finite dimensional vector space over F.

Since R' is not a field, it has a maximal ideal $\mathfrak{m}'$ that is not zero. The ideal $\mathfrak{m}' \cap R$ is not zero (Proposition 28), so R is not a field. Thus every maximal ideal of R is nonzero.

Let $\mathfrak{m}$ be a maximal ideal of R. By the Krull-Akizuki theorem $R'/\mathfrak{m}R'$ has finite length as an R-module, so $R'/\mathfrak{m}R'$ is a finitely generated R-module (Lemma 7). By Lemma 26 there are only a finite number of maximal ideals mapping to $\mathfrak{m}$. □

8 Another Theorem of Akizuki

In 1935 Yasuo Akizuik published a proof that every Artinian ring is Noetherian.[2]. In this section we give a proof of this result based on the techniques developed above.

Lemma 37. *In an Artinian ring the zero ideal is the product of prime ideals.*

Proof. I be an ideal that minimal with respect to the property of being a product of prime ideals. As in Proposition 20, the ideal I^2 is also the product of prime ideals so $I^2 = I$.

Suppose I is not zero. Let aR be a minimal principal ideal such aI is nonzero. Since $aI = aII$ we have that aII is not zero. So there is a $b \in I$ with abI not zero. By minimality of aR this means $aR \subseteq abR$. Thus $a = abc$ for some $c \in R$. In other words $a(1 - bc) = 0$.

Suppose $1-bc$ is not a unit. Then we have $1-bc \in \mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Note that $I\mathfrak{p}$ is the product of prime ideals, so $I = I\mathfrak{p}$ (as in Proposition 20). Thus $I \subseteq \mathfrak{p}$. This implies that $b \in \mathfrak{p}$, so $1 = (1-bc)+bc \in \mathfrak{p}$. This is a contradiction. Thus $1-bc$ is a unit.

This forces $a = 0$, but aI is supposed to be nonzero. We get a contradiction. Thus $I = 0$. □

Theorem 38 (Akizuki)**.** *Any Artinian ring R is a Noetherian ring.*

Proof. We argue via the following steps:

- The zero ideal of R is the product of prime ideals: $0 = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ (Lemma 37).
- Given such a factorization $\mathfrak{p}_1 \cdots \mathfrak{p}_k = 0$, let $I_i = \mathfrak{p}_1 \cdots \mathfrak{p}_i$ and let $I_0 = R$. In other words, $I_i = I_{i-1}\mathfrak{p}_i$ if $i > 0$. We get a finite chain
$$0 = I_k \subseteq I_{k-1} \subseteq \cdots \subseteq I_1 \subseteq I_0 = R.$$
- Each I_i is an Artinian R-module since it is a submodule of R, and R is assumed to be Artinian. (Lemma 2)
- Each quotient I_i/I_{i+1} is the quotient of a Artinian R-module, so must be a Artinian R-module. (Lemma 2)
- Since $\mathfrak{p}_{i+1}$ annihilates $I_i/I_{i+1} = I_i/\mathfrak{p}_{i+1}I_i$, the R-module I_i/I_{i+1} is an $R/\mathfrak{p}_{i+1}$-module. (Lemma 4)

[2]My source for the history of this result is Section 3 of H. Matsumura, *Commutative ring theory*. According to Matsumura, in 1939 C. Hopkins rediscovered this results and generalized it to noncommutative rings, and so it is sometimes called *Hopkins theorem*.

- The quotient I_i/I_{i+1} is a Artinian $R/\mathfrak{p}_{i+1}$-module since it is an Artinian R-module. (Lemma 4)
- Every prime ideal of R is maximal, so $R/\mathfrak{p}_{i+1}$ is a field. Thus I_i/I_{i+1} is a vector space over the field $R/\mathfrak{p}_{i+1}$.
- Each quotient I_i/I_{i+1} is a finite dimensional vector space over the field $R/\mathfrak{p}_{i+1}$ since it is an Artinian $R/\mathfrak{p}_{i+1}$-module. In particular, I_i/I_{i+1} has finite length as an $R/\mathfrak{p}_{i+1}$ module. (Lemma 5)
- Each quotient I_i/I_{i+1} has finite length as an R-module. (Lemma 4)
- $R \cong I_0/I_k$ has finite length as an R-module (Lemma 3).
- Thus R is Noetherian as an R-module (Lemma 1).

□

By combining this theorem with Corollary 19 and Theorem 22 we obtain the following characterization of Artinian rings.

Corollary 39. *Let R be a commutative ring. Then R is an Artinian ring if and only if R is a Noetherian ring and every prime ideal is maximal.*