Chains and Lengths of Modules

A mathematical essay by Wayne Aitken*

Fall 2019†

In this essay begins with the descending and ascending chain conditions for modules, and more generally for partially ordered sets. If a module satisfies both of these chain conditions then it has an invariant called the *length*. This essay will consider some properties of this length.

The first version of this essay was written to provide background material for another essay focused on the Krull–Akizuki theorem and on Akizuki's other theorem that every Artinian ring is Noetherian.[1] The current essay considers Noetherian and Artinian modules, the follow-up essay considers Noetherian and Artinian rings.

I have attempted to give full and clear statements of the definitions and results, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Often supplied proofs are sketches, but I have attempted to be detailed enough that the reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant.

1 Required background

This document is written for readers with some basic familiarity with abstract algebra including some basic facts about rings (at least commutative rings), ideals, modules over such rings, quotients of modules, and module homomorphisms and isomorphisms. For example, the reader should know the correspondence between submodules of a quotient modules M/N and modules X with $N \subseteq X \subseteq M$.

In this document all rings will be commutative with a unity element called 1. We will require that all ring homomorphisms map 1 to 1, and we will require that the element designated 1 in a subring be equal to the element designated 1 in the containing ring.

†Version of December 12, 2019.

[1]See my essay *The Krull–Akizuki Theorem*.

In Propositions 10 and 20, the reader is expected to be familiar with short exact sequences, but these propositions can be skipped if desired.

2 Chains conditions

We formulate the notion of the ascending chain condition (ACC) and the dual notion of the descending chain condition (DCC) in the general context of partially ordered sets.

Definition 1. A *reflexive partial order* on a set S is a relation $\leq$ on S such that (1) the relation is reflexive, (2) the relation is transitive, (3) for all $x, y \in S$ if $x \leq y$ and $y \leq x$ then $x = y$.

A *reflexive total order* on a set S is a reflexive partial order on S with the extra property that for all $x, y \in S$ either $x \leq y$ or $y \leq x$.

A *partially ordered set* is a set $\mathcal{C}$ together with a reflexive partial order relation $\leq$ on $\mathcal{C}$. Similarly a *totally ordered set* is a a set $\mathcal{C}$ together with a reflexive total order relation $\leq$.

Remark. So a partially ordered set has two components: (1) a set and (2) an order relation on that set. The set component is called the *underlying set*. Of course, a given set with more than one element can be the underlying set for multiple partial ordered sets. However, when the order relation is understood, it is customary to identify the ordered set with the underlying set. This convention applies to structured sets in general. So, for example, the term $\mathbb{Z}$ can refer, depending on context, to the set of integers, the ring of integers, the additive group of integers, or the ordered set of integers with the usual order relation. In this section we will typically use the term $\mathcal{C}$ to refer to a partially ordered set and the underlying set.

Remark. Given a reflexive partial or total order $\leq$ on a set $\mathcal{C}$, we can define an associated strict order $<$ on $\mathcal{C}$. It is the relation defined by the rule that $x < y$ if and only if $x \neq y$ and $x \leq y$. It is anti-reflexive ($\neg(x < x)$), transitive, and satisfies the partial trichotomy law: for each pair $x, y \in S$ at most one occurs: $x < y, y < x$, or $x = y$. If $\leq$ is a total order, then $<$ satisfies the total trichotomy law: for each pair $x, y \in S$ exactly one occurs: $x < y, y < x$, or $x = y$.

Given a partially (or totally ordered) set we define the relations $\geq$ and $>$ on that set in the usual way.

Proposition 1. *A subset of a partially ordered set is a partially ordered set, using the induced relation. A subset of a totally ordered set is a totally ordered set, using the induced relation.*

Definition 2. Let S be a nonempty subset of a partially ordered set $\mathcal{C}$. A *maximal* element of S is an element $M \in S$ such that if $M \leq X$ with $X \in S$ then $M = X$. A *maximum* element of S is an element $M \in S$ such that $X \leq M$ for all $X \in S$. We define *minimal* and *minimum* elements in a similar manner.

Remark. A nonempty subset S of a partially ordered set $\mathcal{C}$ does not necessarily have a maximal or maximum element. It can also have multiple maximal elements. A maximal element is not necessarily a maximum. However, a maximum element is necessarily maximal. In fact, if S has a maximum element it is the unique maximum and maximal element of S.

If S is a nonempty totally ordered subset then a maximal element of S, if it exists, is necessarily the maximum of S.

Similar remarks apply to minimal and minimum elements.

Remark. In the context of this document, an *integer index set* is a subset $\mathcal{I}$ of $\mathbb{Z}$ with the property that if $a, b \in \mathcal{I}$ then any $x \in \mathbb{Z}$ with $a < x < b$ is in $\mathcal{I}$. If $\mathcal{I}$ is such an integer index set and if S is a set, then a *sequence* in S indexed by $\mathcal{I}$ is just a function $\mathcal{I} \to S$. We employ the notation $(s_i)_{i \in \mathcal{I}}$ for such a sequence where s_i is understood to be the image of $i \in \mathcal{I}$. We write (s_i) if the index set does not need to be specified.

Definition 3. Let $\mathcal{C}$ be a partially ordered set. An *ascending chain* in $\mathcal{C}$ is a sequence $(C_i)_{i \in \mathcal{I}}$ in $\mathcal{C}$ such that

$$\cdots \le C_{i-1} \le C_i \le C_{i+1} \le \cdots$$

In other words, for all $i, j \in \mathcal{I}$ such that $i < j$ we have $C_i \le C_j$. A *strict ascending chain* is an ascending chain $(C_i)_{i \in \mathcal{I}}$ such that

$$\cdots < C_{i-1} < C_i < C_{i+1} < \cdots$$

We define *descending chains* and *strict descending chains* in a similar manner using the relations $\ge$ and $>$. We use the term *chain* for any sequence (C_i) of elements of $\mathcal{C}$ which forms an ascending or descending chain.

A chain is *bounded* if the index set $\mathcal{I}$ has maximum element. (Typically, $\mathcal{I}$ will have minimum, and the minimum is often 0. But we ignore lower bounds here.) A chain (C_i) *stablizes* if it is either bounded, or if there is a $i \in \mathcal{I}$ such that $C_j = C_i$ for all $j \ge i$).

Remark. Let $\mathcal{C}$ be a partially ordered set. The set of elements occurring in a chain forms a totally ordered subset of $\mathcal{C}$.

Proposition 2. *Let $\mathcal{C}$ be a partially ordered set. Then the following are equivalent:*

1. *Every ascending chain in $\mathcal{C}$ stabilizes.*
2. *Given an unbounded ascending chain $(C_i)_{i \in \mathcal{I}}$ there are only a finite number of positive $i \in \mathcal{I}$ with $C_i < C_{i+1}$.*
3. *Every strict ascending chain in $\mathcal{C}$ is bounded.*
4. *Every nonempty subset of $\mathcal{C}$ contains a maximal element. In fact, given $X \in S$ where S is a subset of $\mathcal{C}$, there is a maximal element M of S with $X \le M$.*

Proof. It is straightforward to prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). $\square$

Definition 4. We say that a partially ordered set $\mathcal{C}$ satisfies the *ascending chain condition* (ACC) if the equivalent conditions of the above proposition hold.

Similarly, we have the following for descending chains and minimal elements:

Proposition 3. *Let $\mathcal{C}$ be a partially ordered set. Then the following are equivalent:*

1. *Every descending chain in $\mathcal{C}$ stabilizes.*
2. *Given an unbounded descending chain $(C_i)_{i\in\mathcal{I}}$ there are only a finite number of positive $i \in \mathcal{I}$ with $C_i > C_{i+1}$.*
3. *Every strict descending chain in $\mathcal{C}$ is bounded.*
4. *Every nonempty subset of $\mathcal{C}$ contains a minimal element. In fact, given $X \in S$ where S is a subset of $\mathcal{C}$, there is a minimal element m of S with $X \geq m$.*

Definition 5. We say that a partially ordered set $\mathcal{C}$ satisfies the *descending chain condition* (DCC) if the equivalent conditions of the above proposition hold.

Proposition 4. *Suppose $\mathcal{C}'$ is a subset of a partially ordered set $\mathcal{C}$. If $\mathcal{C}$ satisfies the ACC then so does $\mathcal{C}'$. If $\mathcal{C}$ satisfies the DCC then so does $\mathcal{C}'$.*

Definition 6. Let $\mathcal{C}$ be a partially ordered set. A *finite chain* is a chain $(C_i)_{\imath\in\mathcal{I}}$ such that the integer index set $\mathcal{I}$ is finite. By reversing the order of the terms C_i if necessary and shifting the indexing if necessary, we can write such a chain as

$$C_0 \leq C_1 \leq \cdots \leq C_n$$

for a unique n, which we call the *length* of the chain. If a finite chain is a strict ascending or descending chain, we call it a *strict finite chain.*

Definition 7. Suppose that $\mathcal{C}$ is a partially ordered set and suppose $X, Y \in \mathcal{C}$ with $X \leq Y$. A *composition series* between X and Y is a strict chain

$$X = C_0 < C_1 < \cdots < C_n = Y$$

that is maximal in the following sense: for $i \in \{1, \ldots, n\}$ there are no $Z \in \mathcal{C}$ such that $C_{i-1} < Z < C_i$.

Proposition 5. *Suppose both the ACC and DCC hold for a partially ordered set $\mathcal{C}$. If $X, Y \in \mathcal{C}$ with $X \leq Y$, then there is a composition series between X and Y.*

Proof. Let $C_0 = X$. We continue recursively as follows: given a term C_i, if $C_i < Y$ then let C_{i+1} be a minimal $D \in \mathcal{C}$ such that $C_i < D \leq Y$. For some i it will happen that $C_i = Y$ (otherwise the ACC would be violated). The resulting chain can be seen to be a composition series between X and Y. $\square$

3 Noetherian and Artinian modules

Let R be a commutative ring. Observe that if M is an R-module, then the collection of submodules of M forms a partially ordered set under the inclusion relation.

Definition 8. An R-module is said to be *Noetherian* if the collection of submodules of M satisfies the ACC. An R-module is said to be *Artinian* if the collection of submodules of M satisfies the DCC.

Although the current essay is focused on modules, we make a brief mention of Noetherian and Artinian rings. Recall that R is itself an R-module.

Definition 9. The ring R is a *Noetherian ring* if R is a Noetherian R-module. The ring R is an *Artinian ring* if R is a Artinian R-module.

In other words, the ring R is Noetherian if and only if the collection of ideals of R satisfy the ACC. Similarly, the ring R is Artinian if and only if the collection of ideals of R satisfy the DCC.

Proposition 6. *Suppose M and M' are isomorphic R-modules. If M is Noetherian then so is M'. If M is Artinian then so is M'.*

Lemma 7. *Suppose M is an R-module. If M is Noetherian then so is any submodule of M. If M is Artinian then so is any submodule of M.*

Lemma 8. *Suppose M is an R-module with submodule N. If M is Noetherian then so is M/N. If M is Artinian then so is M/N.*

Proposition 9. *Suppose M is an R-module with submodule N. Then M is Noetherian if and only if both N and M/N are Noetherian. Similarly M is Artinian if and only if both N and M/N are Artinian.*

Proof. One implication (of both claims) is provided by the previous two lemmas. We will give the other implication in the Noetherian case since the Artinian case is similar. So suppose N and M/N are Noetherian. Consider an ascending chain (M_i) of submodules of M. We can map this into an ascending chain (M'_i) of submodules of M/N that stabilizes for indices $j \geq N_1$ for some integer N_1. Likewise the ascending chain $(N \cap M_i)$ stabilizes for indices $j \geq N_2$ for some integer N_2. Let N be the maximum of N_1 and N_2. It is straightforward to show that (M_i) stabilizes for indices $j \geq N$. □

Proposition 10. *Given a short exact sequence of R-modules*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

then M_2 is Noetherian if and only if both M_1 and M_3 are Noetherian. Similarly, M_2 is Artinian if and only if both M_1 and M_3 are Artinian.

Proof. Note that M_1 is isomorphic to a submodule N of M_2, and M_3 is isomorphic to the quotient M_2/N. So the current result is a corollary to the previous proposition. □

For the most part we consider properties that are symmetric with respect to the Noetherian and the Artinian condition for modules. The following breaks that symmetry:[2]

[2]There are non-finitely generated Artinian modules even for $\mathbb{Z}$-modules. Consider the subgroup of the additive Abelian subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of elements represented by a fraction whose denominator is a power of two. Each proper subgroup is generated by the class of $1/2^n$ for some n. From this property it is straightforward to show the DCC. Observe that this $\mathbb{Z}$-module is not finitely generated.

Proposition 11. *Let M be a Noetherian R-module. Then M is a finitely generated R-module.*

Proof. Consider the collection of all finitely generated submodules of M. It has a maximal element M_0 by the ACC. Let $x \in M$. Then $M_0 + xR$ is also a finitely generated submodule of M, so $M_0 = M_0 + xR$ by maximality of M_0. Thus $x \in M_0$. This shows that $M = M_0$, so M is finitely generated. □

4 The length of a module

Let M be a R-module, where R is a commutative ring, and let N be a submodule of M. There are important situations where there is a composition series (Definition 7) of modules intermediate between N and M, and often such composition series or even the lengths of such series are often helpful in relating N and M. This happens, for example, if $N = I$ and $M = J$ are nonzero ideals of $\mathbb{Z}$. More generally this happens for nonzero ideals of any Noetherian ring where every nonzero prime ideal is maximal (such as PIDs or Dedekind domains).[3] Such series are also related to the concept of the norm of an ideal in algebraic number theory.

A composition series

$$N = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

corresponds to a composition series of quotient modules

$$0 = M_0/N \subsetneq M_1/N \subsetneq \cdots \subsetneq M_n/N = M/N.$$

So we can reduce to the situation where the first term is 0. In this case, the length of the composition series gives a nice generalization to the dimension of a vector space: if R is a field then the dimension of a finite dimensional vector space V is just the length of a composition series between the zero space 0 and all of V.

It will turn out that all composition series between two modules have the same length (assuming at least one such series exists). For now, we define the length of a module M as the minimum length among the composition series between 0 and M:

Definition 10. Let M be a module. A *composition series* for M is a composition series between 0 and M, in the sense of Definition 7, in the collection of submodules of M partially ordered by inclusion.[4] In other words, a composition series of M is a strict finite chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

that is maximal in the following sense: for $i \in \{1, \ldots, n\}$ there are no submodules M' of M such that $M_{i-1} \subsetneq M' \subsetneq M_i$. If such a finite composition series exists, then we say that M has *finite length.* The minimal length n of all such composition series for M is called the *length* of M. We write $\mathrm{len}(M)$ or $\mathrm{length}(M)$ for the length of M.

[3] Suppose $I \subseteq J$ are ideals in such a ring R. It turns out that R/J is a Noetherian and Artinian ring (see, for example, my essay *The Krull–Akizuki Theorem*). Thus I/J is Noetherian and Artinian as an R/J-modules, and hence as an R-module. So, I/J has a composition series.

[4] Given a finite group G we also define composition series (G_i) in terms of the partially ordered collection of subgroups. However, in that case we add the requirement that G_i is a normal subgroup of G_{i+1}.

One fruitful approach to studying composition series is via the Jordan-Holder theorem which we will consider in an appendix. Here we derive the basic properties of length without using the Jordan-Holder approach.

Definition 11. A *simple module* is a nonzero module M whose only submodules are 0 and M.

Proposition 12. *A strict finite chain of submodules of* M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

is a composition series if and only if M_i/M_{i-1} *is simple for each* $i \in \{1, \ldots, n\}$.

Proposition 13. *Suppose* M_1 *and* M_2 *are isomorphic modules. If* M_1 *has finite length, then so does* M_2*, and both modules have the same length.*

Proposition 14. *Let* N *be a submodule of a module* M*. If* M *has finite length then* N *has finite length and* $\text{length}(N) \leq \text{length}(M)$*. If* N *is a proper submodule of* M *then* $\text{length}(N) < \text{length}(M)$.

Proof. Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ be a composition series of minimal length and let $N_i = M_i \cap N$. For each $0 < i \leq n$, consider the composition of the inclusion map followed by the quotient homomorphism: $N_i \to M_i \to M_i/M_{i-1}$. The kernel is N_{i-1}, so there is an injection $N_i/N_{i-1} \to M_i/M_{i-1}$. Thus N_i/N_{i-1} is isomorphic to its image in M_i/M_{i-1}. Since M_i/M_{i-1} is simple, either N_i/N_{i-1} is isomorphic to M_i/M_{i-1}, hence is simple, or N_i/N_{i-1} is isomorphic to 0, in which case $N_i = N_{i-1}$. We have a finite chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$$

but it might not be strict. By repeatedly removing a term N_i with $N_i = N_{i-1}$, if such exists, we eventually form a strict finite chain of length at most n. Since each quotient N_i/N_{i-1} is simple for such a chain, it is a composition series. Thus $\text{length}(N) \leq \text{length}(M)$.

Suppose N_i/N_{i-1} is isomorphic to M_i/M_{i-1} for each i. We claim that $N_i = M_i$ for each i in this case. We prove this by induction so we assume $N_{i-1} = M_{i-1}$ for some $i > 0$. Let $x \in M_i$. Then in M_i/M_{i-1} we have $[x] = [y]$ for some $y \in N_i$. Thus $x = y + z$ for some $z \in N_{i-1} = M_{i-1}$. Thus $x \in N_i$.

So if N_i/N_{i-1} is isomorphic to M_i/M_{i-1} for each i then $N = M$. Thus if $N \subsetneq M$ then the composition series (N_i) for N has strictly smaller length than the composition series (M_i) for M. □

Proposition 15. *Suppose* M *is a module of finite length. Given a strict finite chain of submodules of* M

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k$$

then $k \leq \text{length}(M)$.

Proof. By the previous proposition each M_i has finite length and

$$0 < \text{len}(M_1) < \cdots < \text{len}(M_k) \leq \text{len}(M).$$

□

Proposition 16. *Suppose M is a module of finite length. Given a strict finite chain of submodules of M*

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k$$

Then $k = \text{length}(M)$ if and only if this sequence is a composition series for M. In particular, all composition series have the same length.

Proof. Suppose $k = \text{length}(M)$. If the given chain is not a composition series we could form a longer strict chain of length $k+1$, contradicting Proposition 15. Thus the given chain is a composition series.

Conversely, suppose the given sequence is a composition series. Then we have $k \geq \text{len}(M)$ by definition of length. However, $k \leq \text{length}(M)$ by Proposition 15. Thus $k = \text{length}(M)$. □

Proposition 17. *A module M has finite length if and only if it is both Noetherian and Artinian.*

Proof. If M is both Noetherian and Artinian, then it has a composition series (Proposition 5). Now suppose M has finite length. All strict ascending or descending chains of submodules of M must be finite by Proposition 15. □

Proposition 18. *A module M fails to have finite length if and only if for all $n \in \mathbb{N}$ there is a strict finite chain of submodules of M*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n.$$

Proof. If M fails to have finite length then there is no composition series, which means that any strict chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k$ can be expanded to a strict chain of length $k+1$. By starting with $k = 0$, we can grow the chain to any desired length.

Conversely, suppose M has finite length k. Then any such sequence has length bounded by k(Proposition 15). □

We can strengthen Proposition 14:

Proposition 19. *Suppose that M is an R-module and that N is an R-submodule of M. Then M has finite length if and only if N and M/N both have finite length. In this case,*

$$\text{length}(M) = \text{length}(N) + \text{length}(M/N).$$

Proof. We combine Proposition 9 and Proposition 17 to conclude that M has finite length if and only if N and M/N both have finite length.

Now suppose N and M/N, and so M, are all of finite length. Let

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = N$$

be a composition series of N, and let

$$0 = M_0/N \subsetneq M_1/N \subsetneq \cdots \subsetneq M_l/N = M/N$$

be a composition series of M/N. Then

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_l = M$$

is a composition series for M. The length of a module is the length of any composition series, so the additivity follows. $\square$

Proposition 20. *Suppose* $0 \to M_1 \to M \to M_2 \to 0$ *is an exact sequence of modules. Then* M *has finite length if and only if both* M_1 *and* M_2 *have finite length. In this case*

$$\text{length}(M) = \text{length}(M_1) + \text{length}(M_2).$$

Proof. Observe that M_1 is isomorphic N where N is the kernel of $M_1 \to M_2$. Observe that M_2 is isomorphic to M/N. Thus the result follows from the previous proposition. $\square$

Proposition 21. *If* M *is an* R*-module of finite length. Then* M *is a finitely generated* R*-module.*

Proof. If M is of finite length then it is Noetherian (Proposition 17). So M is finitely generated (Proposition 11). $\square$

Proposition 22. *Suppose* M *is an* R*-module and suppose* I *is an ideal of* R *that annihilates* M *in the sense that* $aM = 0$ *for each* $a \in I$. *Then* M *is naturally an* R/I*-module. In this case an Abelian subgroup* N *of* M *is an* R*-submodule of* M *if and only if* N *is an* R/I*-submodule of* M.

Conversely if M *is an* R/I*-module for some ideal* I *of* R *then* M *is naturally an* R*-module and* I *annihilates* M.

Corollary 23. *Suppose* M *is an* R*-module and suppose* I *is an ideal of* R *that annihilates* M *in the sense that* $aM = 0$ *for each* $a \in I$. *Then any ascending chain of* R*-submodules of* M *is an ascending chain of* R/I*-submodules of* M *and vice versa. Any descending chain of* R*-submodules of* M *is a descending chain of* R/I*-submodules of* M *and vice versa. A composition series for* M *as an* R*-module is a composition series for* M *as an* R/I*-module, and vice versa.*

So M *is Noetherian as an* R*-module if and only if it is Noetherian as an* R/I*-module. Likewise,* M *is Artinian as an* R*-module if and only if it is Artinian as an* R/I*-module. Similarly,* M *has finite length as an* R*-module if and only if it has finite length as an* R/I*-module. In this case the lengths are the same (as an* R*-module versus an* R/I*-module).*

Proposition 24. *Let* M *be an* F*-vector space where* F *is a field. Then* M *has finite length if and only if* M *is a finite dimensional* F*-vector space. In this case the length is just the dimension. In fact, for such vector spaces* M *the following are equivalent:*

1. *M has finite length.*
2. *M has finite dimension.*
3. *M is Noetherian.*
4. *M is Artinian.*

Proposition 25. *Let M be an R module, and let*

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k$$

be a finite chain of submodules of M. Then M_k/M_0 has finite length if and only if M_i/M_{i-1} has finite length for each $i \in \{1, \ldots, k\}$. In this case,

$$\operatorname{len}(M_k/M_0) = \operatorname{len}(M_k/M_{k-1}) + \ldots + \operatorname{len}(M_2/M_1) + \operatorname{len}(M_1/M_0).$$

Proof. We prove this by induction on k. The case $k = 1$ is immediate, so we consider the case $k > 1$. We have

$$M_{k-1}/M_0 \subseteq M_k/M_0$$

and

$$(M_k/M_0)/(M_{k-1}/M_0) \;\cong\; M_k/M_{k-1}.$$

We now use Proposition 19. □

5 Simple modules

We will now consider some results about simple modules over a commutative ring R.

Using the correspondence between submodules of M/N and submodules of M containing N we get the following.

Lemma 26. *Let I be an ideal of a commutative ring R. Then R/I is a simple R-module if and only if I is a maximal ideal.*

Lemma 27. *Let $\varphi\colon M_1 \to M_2$ be an injective homomorphism between R-modules. If M_2 is simple and if M_1 is nonzero then φ is an isomorphism and so M_1 is simple.*

Lemma 28. *Let M be a simple R-module. Then M is isomorphic as an R-module to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.*

Proof. Fix $x \in M$ a nonzero element of M and consider the function $r \mapsto rx$. This map is an R-module homomorphism $R \to M$. If I is the kernel then there is an injective homomorphism $R/I \to M$. Since M is simple and since $I \neq R$ (since $1 \notin I$), this map is an isomorphism (Lemma 27). Thus R/I is simple as an R-module, and so I is a maximal ideal of R (Lemma 26). □

Proposition 29. *Let M be a module over a commutative ring R. Then the following are equivalent.*

1. *M is simple.*
2. *M has has finite length over R with* $\operatorname{length}(M) = 1$.
3. *M is isomorphic, as R-modules, to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R.*

6 Appendix: the Jorden-Holder approach

We will given another approach to the lengths of modules that also considers the quotients that appear. This is the Jorden-Holder approach. Given two composition series

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M, \qquad 0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_l = M$$

the Jorden-Holder approach will not only show that $k = l$, but will also show that, up to permutation and isomorphism, the simple quotients M_i/M_{i-1} of the first composition series are the same as the simple quotients M'_i/M'_{i-1} of the second composition series.

Throughout this appendix, we adopt the terminology and methodology of multisets of modules up to isomorphism. To make this more precise, we will define multisets as a kind of equivalence class. Given sequences $(X_i)_{i\in\mathcal{I}}$ and $(X'_i)_{i\in\mathcal{I}'}$ of R-modules indexed by finite sets $\mathcal{I}$ and $\mathcal{I}'$, we say that $(X_i)_{i\in\mathcal{I}}$ and $(X'_i)_{i\in\mathcal{I}'}$ are equivalent if there is a bijection $\sigma : \mathcal{I} \to \mathcal{I}'$ such that X_i is isomorphic to $X'_{\sigma i}$ for all $i \in \mathcal{I}$. This is clearly an equivalence relation between such sequences.[5] We view a *multiset* of modules up to isomorphism to be an equivalence class of this equivalence relation. We will use the notation $\{C_i\}$ for the multiset associated with the representative sequence (C_i). The cardinality of a multiset is, of course, the cardinality of an index set of any representative sequence. Given two multisets, we can form a well-defined union which is a multiset whose cardinality is the sum of the cardinalities of the given multset.

Our goal in this appendix is to prove the following:

Theorem 30 (Jorden-Holder for modules)**.** *Let M be a module and let*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M, \qquad 0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_l = M$$

be two composition series. Then $k = l$ and $\{M_i/M_{i-1}\}$ and $\{M'_i/M'_{i-1}\}$ are equal as multisets up to isomorphism.

Our strategy is to set up a two-dimensional array of R-modules. For each i, j with $0 \le i \le k$ and $0 \le j \le l$ let $N_{i,j}$ be the module defined as follows

$$N_{i,j} \stackrel{\text{def}}{=} M_i \cap M'_j.$$

In particular, $N_{0,j} = N_{i,0} = 0$, $N_{i,l} = M_i$, $N_{k,j} = M'_j$, and $N_{k,l} = M$.

Remark. One method of proof can be described informally as follows: For any path from $(0, 0)$ to (k, l), such that each step increases either the first coordinate or the second coordinate by one unit, we can generate a composition series using the $N_{i,j}$ for (i, j) along the path (discarding extra isomorphic terms). The path

$$(0, 0), (0, 1), \ldots, (0, l), (1, l), \ldots, (k, l)$$

yields the first given composition series, and the path

$$(0, 0), (1, 0), \ldots, (k, 0), (k, 1), \ldots, (k, l)$$

[5] We will limit our attention to to quotients of submodules of a fixed module M and we will limit our index sets to subsets of $\mathbb{Z}$ or $\mathbb{Z}^2$. So the collection of such sequences forms a set.

yields the second given composition series. We then show that for each modification of a path by replacing a sequence of steps $(i-1, j-1)$ to $(i-1, j)$ to (i, j) by the sequence $(i-1, j-1)$ to $(i, j-1)$ to (i, j) yields the same length of composition series with the same multiset of simple quotients (using the lemmas below). We then need to argue that any path can be changed to any other by a series of such modifications ("homotopy"). This approach is intriguing, but it involves work to make it formal. So we will pursue another approach involving cancelling like terms in certain multisets.

Either approach will require two key lemmas:

Lemma 31. *Suppose A, B, N are submodules of M such that $A \subseteq B$. Suppose also that B/A is simple. Then either $B \cap N = A \cap N$ or $(B \cap N)/(A \cap N)$ is isomorphic to B/A.*

Proof. Consider the composition of the inclusion followed by the quotient map.

$$B \cap N \to B \to B/A.$$

Observe that the kernel is just $A \cap N$. Thus there is an injection

$$(B \cap N)/(A \cap N) \to B/A.$$

Since B/A is simple, the image is 0 or all of B/A. In the first case $B \cap N = A \cap N$. In the second $(B \cap N)/(A \cap N)$ is isomorphic to B/A. $\square$

Lemma 32. *Let M be a module. Suppose A, B, C, D are submodules of M such that $A \subseteq B$ and $C \subseteq D$. Suppose also that B/A is simple and that D/C is simple. Then we have*

$$\{B \cap D/B \cap C,\ B \cap C/A \cap C\} = \{B \cap D/A \cap D,\ A \cap D/A \cap C\}$$

as multisets up to isomorphism.

Proof. Suppose first that $(B \cap D)/(A \cap C)$ is simple. Since $B \cap C$ and $A \cap D$ are each intermediate between $A \cap C$ and $B \cap D$, each are equal to $A \cap C$ or $B \cap D$. Thus, each multiset is $\{0, (B \cap D)/(A \cap C)\}$, so they are equal.

Next suppose that $(B \cap D)/(A \cap C)$ is not simple. By the previous lemma, each element of each multiset is either 0 or simple. Since $(B \cap D)/(A \cap C)$ is not simple, no element can be 0. Since each element is simple, we have, by the previous lemma

$$\{B \cap D/B \cap C,\ B \cap C/A \cap C\} = \{D/C, B/A\}$$

and

$$\{B \cap D/A \cap D,\ A \cap D/A \cap C\} = \{B/A, D/C\}.$$

$\square$

Corollary 33. *let i, j be such that $1 \leq i \leq k$ and $1 \leq j \leq l$. Then as multisets up to isomorphism*

$$\{N_{i,j}/N_{i-1,j},\ N_{i-1,j}/N_{i-1,j-1}\} = \{N_{i,j}/N_{i,j-1},\ N_{i,j-1}/N_{i-1,j-1}\}.$$

Now we can give an argument for the key result. We set-up notation for two types of quotients of the N_{ij}: For each $0 \le i \le k$ and $1 \le j \le l$ define

$$H_{i,j} \stackrel{\text{def}}{=} N_{i,j}/N_{i,j-1} = (M_i \cap M'_j)/(M_i \cap M'_{j-1}).$$

For each $1 \le i \le k$ and $0 \le j \le l$ define

$$V_{i,j} \stackrel{\text{def}}{=} N_{i,j}/N_{i-1,j} = (M_i \cap M'_j)/(M_{i-1} \cap M'_j).$$

So the above corollary states that $X_{i,j} = Y_{i,j}$ where

$$X_{i,j} \stackrel{\text{def}}{=} \{V_{i,j}, H_{i-1,j}\}, \quad Y_{i,j} \stackrel{\text{def}}{=} \{H_{i,j},\ V_{i,j-1}\}.$$

Now define two multisets X and Y (each of cardinality $2kl$):

$$X \stackrel{\text{def}}{=} \bigcup_{\substack{1\le i\le k\\ 1\le j\le l}} X_{i,j}, \qquad Y \stackrel{\text{def}}{=} \bigcup_{\substack{1\le i\le k\\ 1\le j\le l}} Y_{i,j}.$$

By the above corollary, $X = Y$. Now consider the multiset W:

$$W \stackrel{\text{def}}{=} \{H_{i,j} \mid 1 \le i \le k-1, 1 \le j \le l\} \cup \{V_{i,j} \mid 1 \le i \le k, 1 \le j \le l-1\}$$

Note that

$$X = W \cup \{H_{0,1}, H_{0,2}, \ldots, H_{0,l}, V_{1,l}, V_{2,l}, \ldots, V_{k,l}\}$$

and

$$Y = W \cup \{H_{k,1}, H_{k,2}, \ldots, H_{k,l}, V_{1,0}, V_{2,0}, \ldots, V_{k,0}\}.$$

Since $X = Y$, this means (as multisets up to isomorphism of size $k + l$)

$$\{H_{0,1}, \ldots, H_{0,l}, V_{1,l}, V_{2,l}, \ldots, V_{k,l}\} = \{H_{k,1}, H_{k,2}, \ldots, H_{k,l}, V_{1,0}, \ldots, V_{k,0}\}.$$

In terms of M_i and M'_i this gives multisets

$$\{0, \ldots, 0, M_1/M_0, \ldots, M_k/M_{k-1}\} = \{M'_1/M'_0, \ldots, M'_k/M'_{k-1}, 0, \ldots, 0\}$$

where the left side has 0 occurring l-times and the right side has 0 occurring k-times. Thus $k = l$. Removing the 0 terms gives the desired equality of multisets.

Remark. This proof can be adapted to the composition series of finite groups, which yields the Jordan-Holder theorem for finite groups.