Bourbaki, Theory of Sets, Chapter I, *Description of Formal Mathematics*: Summary and Commentary

Commentary Wayne Aitken*

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These are my notes on Bourbaki's chapter called *Description of Formal Mathematics*. Logically speaking *Description of Formal Mathematics* is the first chapter of Bourbaki's *Elements of Mathematics*, but was not the first to be published. The first publication by Bourbaki was a short summary of results in set theory (1939) followed by the topology and algebra chapters (largely in the 1940s). The chapter under consideration here was published around 1954 originally.[1]

This first chapter of Bourbaki is about 50 pages in the original including exercises.

This document is a detailed summary and commentary of this chapter of Bourbaki. I have modernized the original while remaining faithful to the spirt of the original. This means that I have updated some of the terminology, changed some proofs and made some changes in the order of the results. This includes adding some auxiliary results when I thought that they added value or helped clarify the original. For example, I have made extensive use of the technique of introduction and elimination (meta)theorems in this exposition which is a hallmark of natural deduction approach. I have also tried to unify the syntactic results related to quantifiers and bound variables (See Section 10.1 below).

This is a commentary in the sense that I sometimes give explicit comments, but also more subtly by my choices in changing terminology, phrasing, notation, or proof in an effort to clarify Bourbaki for myself and a modern reader. So to get the full Bourbaki experience, the reader should read this document alongside Bourbaki's original. But if the reader just wants a good sense of what is in Bourbaki this document is reasonably complete. So this document can potentially be of service to two classes of readers: (1) readers of Bourbaki's original first chapter who want to benefit from extra commentary and some indication on how the approach looks from a more modern point of view, and (2) readers who, without necessarily reading Bourbaki's chapter, want to get a feel for Bourbaki's approach, but from a more modern point of view.

[1] Since Bourbaki is a group of authors, I will sometimes regard "Bourbaki" as a plural noun. Often I will regard it as a singular noun as well since Bourbaki themselves did so. Which option I choose is a matter of emphasis.

1 General comments about *Elements of Mathematics*

The Bourbaki series *Elements of Mathematics* "takes up mathematics at the begining, and gives complete proofs". "In principle, it requires no particular knowledge of mathematics on the reader's part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought" [pg. v].

I would not describe these books as a series of introductory textbooks, nor would I describe them as an encyclopedia of mathematics. These books are not introductory since they follows the order of development imposed by the logical structure of the proofs and their structural and axiomatic approach to the subject matter, and not a conceptual or historical order designed for accessiblity. It is not an encyclopedia since the coverage is limited to foundational and settled material, with not much attention to more applied topics *per se* (although the ideas presented are certainly central to mathematics as a whole). What Bourbaki wrote can be viewed as a significant and extensive foundational series, supplemented with good exercises for students and useful historical notes. Its "main purpose" is "to provide a solid foundation for the whole body of modern mathematics" [pg. v]. It does have a limitation as a full foundation, at least from today's point of view, simply because it predates category theory. Its avoidance of category theory is a significant gap. Although Bourbaki members were influential in the development of category theory, they choose not to rewrite the *Elements of Mathematics* from a categorical point of view but rather to expand beyond the earlier work.

I have studied *Elements of Mathematics* largely through the English translations published by Springer. The chapter under consideration is the first chapter in *The Theory of Sets* published in 2004 by Springer. It is based on the 1970 French edition, *Éléments de Mathématique, Théorie des Ensembles.* My page references are to this 2004 edition.

It seems to me that the *Elements* can naturally be divided into 3 parts. The first part contains chapters related to set theory, algebra, and general topology. Theory of sets is covered in 4 chapters, algebra in 7 chapters in English (at least 10 chapters in the French edition) and general topology in 10 chapters. I view these first 21 chapters (in English) as the first part. Next comes various aspects of analysis (another 20 or so chapters). This is followed by a third part: additional material in Lie theory, commutative algebra, and spectral theories (20 chapters or so). Overall there are around 60 chapters; this document concerns just the first chapter, which is much different than most of the others in style and subject matter.

2 Introduction to formalization

For Bourbaki formalization is a means to achieve an objective standard for rigorous proof. Formalization is also used by mathematical logicians to aid in consistency proofs and as a tool for other metamathematical investigations, but these are less central to Bourbaki's project.

Bourbaki is not concerned so much with certainty, but is rather more concerned with rigor and its use in the service of the axiomatic method. In particular, they do not view the axioms as absolutely certain, and if they turn out to be inconsistent they are confident that they could be replaced by other axioms which would provide a foundation for most of modern mathematics.

2.1 Standard of rigorous proof

Bourbaki uses the concept of a formalized proof as a standard for a community judgement of validity of a proof. Well established principles of mathematical proof can be codified and formalized, especially in light of the 20th century formalization of mathematical logic. In fact enough principles can be formalized that experience has shown that all rigorous proofs can be expressed using a sufficient set of formal principles, at least in principle. The potential for a proof to be so formalized gives an objective standard of proof. For a fully formalized proof, checking its validity relative to such formal rules is a mechanical process, and a good computer program could check the details. So it is possible for many theorems to be judged to have a complete proof using such software. Only select theorems, however, will likely be so formalized due the effort involved. In practice, however, proofs that have not been so formalized can still be regarded as valid and rigorous if judged to be *formalizable*. Of course, this is a judgement call – but detailed proofs which do not depend on appeal to intuitions can and do achieve this standard. The results in the series *Elements of Mathematics*, for example, do achieve this standard, at least in the judgement of Bourbaki. Bourbaki describes this ideal as follows [pg. 8]:

> In practice, the mathematician who wishes to satisfy himself of the perfect correctness or "rigour" of a proof or a theory hardly ever has recourse to one or another of the complete formalizations available nowadays, nor even usually to the incomplete and partial formalizations provided by algebraic or other calculi. In general he is content to bring the exposition to a point where his experience and mathematical flair tell him that translations into formal languages would be no more than an exercise in patience (thought doubtless a very tedious one). If, as happens again and again, doubts arise as to the correctness of the text under consideration, they concern ultimately the possibility of translating it unambiguously into such a formalized language : either because the same word has been used in different senses according to the context, or because the rules of syntax have been violated by the unconscious use of modes of argument which they do not specifically authorize, or again because of a material error has been committed. Apart from this last possibility, the process of rectification, sooner or later, invariably consists in the construction of texts which come closer and closer to a formalized text until, in the general opinion of mathematicians, it would be superfluous to go any further in this direction. In other words, the correctness of a mathematical text is verified by comparing it, more or less explicitly, with the rules of a formal language.

This is not necessarily a good description of *actual* mathematical practice of judging the completeness and rigor of a proof, but it certainly communicate Bourbaki's notion of formalization as a standard of proof. Of course formalizing proof requires formalizing the language of mathematics. So a formal language of mathematics is an important aspect of the project. Since formal verification of formal proofs are to be fully mechanical, the meaning of the formal sentences is not relevant, except as a motivation for axioms and basic rules of inference. In other words, the correctness of a proof can be judged by reference to syntax alone without appeal to the intended semantics, that is the intended interpretation, of the formal language. Once axioms and logical rules are fixed, meaning does not enter into the question of whether or not a proof is formally correct, and checking correctness is a purely mechanical process.

So the aim is not to achieve total certainty, but to achieve a pragmatic approach to rigorous proof. After the first few chapters of the *Elements* the actual level of formality will decrease: "We shall therefore very quickly abandon formalized mathematics, but not before we have carefully traced the paths which leads back to it" [pg. 11]. Most of the series will be written "partly in ordinary language and partly in formulae which constitute partial, particular, and incomplete formalization, the best-known examples of which are the formulae of algebraic calculation" [pg. 11]. For the benefit of the reader, appeal to intuition will be given for purely expository purposes: "this use of the resources of rhetoric is perfectly legitimate, provided only that the possibility of formalizing the text remains unaltered" [pg. 11].

> Thus, written in accordance with the axiomatic method and keeping always in view, as it were on the horizon, the possibility of a complete formalization, our series lays claim to perfect rigor : a claim which is not in the least contradicted by the preceding considerations, nor by the need to correct errors which slip into the text from time to time. [pg. 12]

2.2 Axiomatic method

As alluded in the above quote, the idea of formalizability is also the basis of Bourbaki's conception of the axiomatic method. Bourbaki notes that the (rigorous) axiomatic method is critical to modern mathematics: "its systematic use as an instrument of discovery is one of the original features of contemporary mathematics" [pg. 8]. This is because the meaning of the language does not enter into the judgement of the validity of a formalizable proof. So the proof remains valid under suitable changes of meaning of the terms. This is analogous to the fact that the "same algebraic calculation can be used to solve problems about points weight or pounds sterling, about parabolas or motion under gravity" [pg. 8]. They also regard the axiomatic method as a good way to separate and organize properties. For example, separating algebraic properties from topological properties.

So Bourbaki's view of the axiomatic method differs from the more traditional conception of axioms that is tied to intuition. Traditionally the axioms are viewed as being evidently true of objects with a fixed interpretation in view. For example, the axioms of geometry are viewed as applying just to geometric objects. The axioms of the natural numbers (Peano's axioms), or Hilbert's full list of axioms of geometry are more traditional in spirit since there is essentially one model for such axioms. The axioms of a topological space, or partial collections of the Hilbert axioms of geometry are more in the spirit of Bourbaki's axiomatic method since there are multiple models, where results inspired by some interpretations will hold in all interpretations and so results proved in one context can be transported to other contexts. So the key to Bourbaki's conception of the axiomatic method is the idea of a purely syntactical conception of proof, where one's intuitions will not enter into the judgement of the correctness of a proof and so where proofs can be transferable to new interpretations of the axioms.

2.3 Methodology

Bourbaki, in their Chapter I, explains formalization by example. They develop one particular formal language, but the ideas developed here mostly extend to other possible formalizations. To set a standard of proof, and the idea of formalizable proofs, one formal language is enough. They do not think that their language is sacrosanct in any sense, and they believe that other formalizations might be necessary to capture future mathematics: "It may happen at some future date that mathematicians will agree to use modes of reasoning which cannot be formalized in the language described here : it would then be necessary, if not to change the language completely, at least to enlarge its rules of syntax. But that is for the future to decide" [pg. 9]. So any criticism of Bourbaki's particular formal system is not an essential criticism of the project as a whole, since the basic formal level could be swapped out by a more elegant or useful system.

They are pessimistic about the practical issues of achieving full formalization: "no great experience is necessary to perceive that such a project is absolutely unrealizable : the tiniest proof at the beginnings of the Theory of Sets would already require several hundreds of signs for its complete formalization" [pg. 10]. "...formalized mathematics cannot in practice be written down in full, and therefore we must have confidence in what might be called the common sense of the mathematician" [pg. 11]. Full formalization, according to Bourbaki, is not practical, but is useful to set a standard. In practice, various metamathematical arguments are used to generate new rules of inferences and "abbreviating symbols" are brought into play to allow a more flexible language allowing a more condensed form of expression. One can use metamathematical judgements to assure oneself that proofs in this more flexible language are convertible to the original primitive formal language. But there will be some loss of reliability since one's judgements depend on the validity of the metamathematical reasoning used to support the abbreviations and new deductive criteria. So they regard such extended languages as less perfect and reliable, but much more usable.

Here Bourbaki is unnecessarily dismissive and pessimistic. More practical formal languages have emerged that are more workable. Many have been implemented in computer software, and have been extensively used. Part of the problems described by Bourbaki can be cured by good language design (where, for example, the ability to define new symbols or abbreviations is built into the formal language itself). So this part of Bourbaki needs updating: it is possible to fully formalize mathematics in a more user friendly manner.

What preexisting language is used in specifying a formal language of mathematics and mathematical proof? Which preexisting logic and mathematicals is used to reason about the formal language to devise various short-cuts, and check proofs, even abbreviated proofs, for validity? Metamathematical reasoning about symbolic languages is allowed. Formal expressions are treated, officially at least, as assemblages of symbols, ignoring "any meaning which may originally have been attributed to the words or phrases of formalized mathematical texts" [pg. 10]. Simple combinatorial reasoning is allowed at this meta-level. Finitary forms of reasoning, including induction, are allowed and sometimes employed, but the level of the metamathematics used to describe the formal language is modest. Numerals are used, but more as "marks" or labels. Heavier metamathematics is used to justify useful deductive criteria that make it easier to write proofs.[2]

2.4 Consistency

Hilbert and others created and refined tools of formalization in order to prove consistency of a theory, where the consistency proof itself is conducted in the metamathematics. However, Gödel showed that you would need to incorporate strong mathematical principles in the metamathematics: for interesting theories the metamathematics cannot be weaker, or even at the same level, as the mathematics you are trying to prove to be consistent.

Bourbaki contrasts this with relative consistency. Here the metamathematics can be kept quite simple. Consistency of various theories can be proved assuming the consistency of formal set theory (this is often what is meant when a theory is proved to be consistent). So the question of the consistency of set theory becomes important. Bourbaki believes that if set theory turns out to be inconsistent in its current form, it can be modified in such a way that important mathematical theories can be kept relatively intact. An example of this is how naive set theory was modified by Zermelo and others without compromising mathematics.

> To sum up, we believe that mathematics is destined to survive, and that the essential parts of this majestic edifice will never collapse as a result of the sudden appearance of a contradiction; but we cannot pretend that this opinion rests on anything more than experience. Some will say that this is a small comfort; but already for two thousand five hundred years mathematicians have been correcting their errors to the consequent enrichment and not impoverishment of their science; and this gives them the right to face the future with serenity. [pg. 13]

[2]Of course using strong mathematics in the metamathematics to justify deductive criteria makes these criteria less certain than the lower level rules. So the level of certainty on what constitutes a valid proof drops a bit when we make use of such criteria to abbreviate our proofs. But note that the deductive criteria will typically be constructive, and will lead to explicit algorithms for transforming an abbreviated proof to a full proof. So in a computerized environment the reliability of the deductive criteria does not result in loss of certainty in concrete cases: one can transform one's abbreviated proof to a full formal proof, and then verify the formal proof by a "trusted kernel". If for some reason a deductive criteria has a flaw that exhibits itself in certain edge cases, it will simply fail to transform some abbreviated proofs into valid formal proofs. So an invalid abbreviated proofo will not mistakenly be judged valid unless the trusted kernel is itself defective.

3 Terms, formulas, and theorems (§1 and §2 in Bourbaki)

I will now describe Bourbaki's formal language. The language itself will be described faithfully, but I will alter some of the meta-mathematical terminology to be more consistent with contemporary descriptions of formal systems, or at least descriptions that I am more comfortable with. For example, I will use the term "formula" rather than "relation". When my terminology differs, I will often indicate Bourbaki's terminology in parentheses, or in other ways. I also make other editorial changes, rearrangements, and so on to make this summary more accessible to a 21st century reader.

Warning: Bourbaki made their language unnecessarily non-user-friendly by using a high level of reductionism, and by using Polish notation and then almost immediately abandoning it. Also using τ as a logical notion restricts the applicability of the logic since it bakes into the logic the axiom of choice; mathematicians who are curious about what mathematics is dependent on the axiom of choice, or what alternatives to this axiom can prove, would not be able to work in this system. So if we were to choose a formalization for the Bourbaki project today, we would likely make different choices to avoid these issues.[3]

3.1 Formal expressions

The alphabet of primitive symbols (Bourbaki: signs) in the primitive language consists of the following.

- Logical symbols (Bourbaki: logical signs): $\square, \tau, \vee, \neg$.
- Variables (Bourbaki: letters): Roman letters: upper and lower case, with one or more prime allowed. $(A, A', A'', \ldots)$.
- Specific symbols of the theory in question (Bourbaki: specific signs). Most theories will use $=$. Set theory adds $\in, \supset$. (Note: the symbol $\supset$ seems to have been removed in later editions. It is used to specify ordered pairs.)

  Each specific symbol has a natural number arity (Bourbaki: weight) with default arity being 2. Furthermore each specific symbol is classified as either "functional" (Bourbaki: substantific) or "relational". We call this designation the *classification* of the specific symbol. The symbol $=$, when used in the theory, will be relational with arity 2. In set theory $=$ and $\in$ are relation symbol with arity 2, while $\supset$ is a function symbol with arity 2. I do not believe Bourbaki allows arity 0, but one could as a way to introduce constants. Arity 2 is usually called "binary" and arity 1 is called "unary".[4]

There are an infinite number of variables, so the alphabet is infinite. This is a potential infinity: in any given formal document only a finite number of variables

[3]Reductionism and extreme simplicity of the language are often virtues in metamathematical contexts. But Bourbaki advocates formalism primarily as a standard of rigor, not for its use in metamathematics. So I do not see the point of excessive reductionism here. I would tend to err on the side of transparency.

[4]The theories constructed by Bourbaki only use arity 2. So one could avoid reference to natural numbers here, and stick to arity 2 only if one wants to limit the metalanguage.

will be in use. But the potential exists at each point to introduce a new variable, if needed, that is not in use yet.

An *expression* (Bourbaki "assembly") is a string of symbols together with zero or more *links*. A link is indicated by drawing lines above the expression connected two symbols occurring in the expression.[5] A link can only be given between symbols of the expression occurring in distinct positions. Bourbaki explicitly forbids variables to occur in a link. Apparently Bourbaki only allows at most one link between any pair of positions in a string, so a link can really be thought of as a choice of two distinct positions in a string of symbols, and Bourbaki's notation using lines above the string of symbols can be regarded as a convenience for this more basic idea. Indeed, Bourbaki abandon's this notation almost immediately and uses various abbreviating expressions instead.

As we will see (in a metaresult) we only really need to allow links between an occurrence of τ and a following occurrence of $\square$ in the expression. In fact, we can just require that an expression must link each occurring $\square$ to one and only one preceding τ symbol, and allow no other links.[6]

We do not really need an empty string here, so I will assume that Bourbaki requires each expression have at least one symbol. (However, it might be handy to have an empty expression in some meta-arguments, and the empty expression could be admitted with no detrimental effects.)

3.2 Metavariables and definitions

Bourbaki uses boldface letters in the metalanguage as syntactic metavariables for symbols or expressions. In addition, primitive symbols such as $\square$ or $\vee$ are used in the metalanguage where they refer to their siblings in the formal language. The metalanguage usage of such primitive symbols, and usage of juxtaposition in the metatheory, follow the usual common sense conventions. In particular, we could write $\tau\mathbf{A}$, say, to refer to the expression obtained by appending τ to the front of the expression $\mathbf{A}$ (i.e., prepending τ), with no new links added. Similarly $\mathbf{AB}$ denotes the expression formed from the juxtaposition of expressions $\mathbf{A}$ and $\mathbf{B}$. The links present in $\mathbf{AB}$ will just be the links present in $\mathbf{A}$ or $\mathbf{B}$: no symbol originally in $\mathbf{A}$ will be linked to a symbol originally in $\mathbf{B}$.

Bourbaki does not incorporate a formal mechanism of definition, but instead regards definitions as occurring in the metalanguage, and are handled in a common sense manner. Such definitions are viewed as conventions to abbreviate or describe formal expressions using more informal and consise language. We start with two such definitions:

Definition 1. If $\mathbf{A}$ is a formal expression and if $\mathbf{x}$ is a variable then $\tau_{\mathbf{x}}(\mathbf{A})$ will denote the expression obtained from $\tau\mathbf{A}$ by replacing each instance of $\mathbf{x}$ with $\square$ and

[5]Links are used as a way to eliminate the need for bound variables in the primitive syntax. Bourbaki will soon go beyond the basic formal language using various conventions and abbreviations, and the less formal language will indicate and use bound variables in the usual manner.

[6]The symbol $\square$ is how Bourbaki denotes a bound variable in the primitive syntax. Most of the other primitive symbols have familiar meanings with a few exceptions: we mentioned that $\subset$ will be used in the definition of ordered pairs (and is not really used until Chapter II, *The Theory of Sets*), and τ, which is related to "choice", will be discussed later. Note the variables are intended to vary over some fixed domain; in the theory of sets, they vary over sets.

by linking each such $\square$ to this leading τ. (All other links are just the links already present in $\mathbf{A}$. Recall that $\mathbf{x}$ is not allowed to be part of a link in any expression.)

Definition 2. If $\mathbf{A}$ and $\mathbf{B}$ are expressions and $\mathbf{x}$ is a symbol, we write $(\mathbf{B}|\mathbf{x})\mathbf{A}$ for the expression created by replacing each occurrence of $\mathbf{x}$ with $\mathbf{B}$. The replacements are done in parallel. (This is important to specify if $\mathbf{B}$ itself has occurrences of $\mathbf{x}$). The links in $(\mathbf{B}|\mathbf{x})\mathbf{A}$ come from the links in $\mathbf{A}$ (which will not involve $\mathbf{x}$), and the links present in $\mathbf{B}$ which will be present in each replacement of $\mathbf{x}$ by $\mathbf{B}$.

Note that we can also view $\tau_{\mathbf{x}}$ and $(\mathbf{B}|\mathbf{x})$ as operators transforming expression to expressions.

3.3 Terms and formulas

Informally, terms are intended to denote objects in the domain of the theory, and formulas are intended to provide sentences or assertions about such objects. More precisely, for every assignment of the occurring variables, a term should denote a certain object and a formula should denote a certain sentence, but the meaning varies with, so is in a sense a function of, the values for the occurring variables.

The above is just informal motivation for the definition. The actual definition is purely syntactic. Note that Bourbaki employs a Polish prefix notation, but then almost immediately replaces this with metalanguage abbreviations that use the familiar infix notation. The advantage of the prefix notation is that parentheses are not needed. Bourbaki regards parentheses as metatheoretic notation used in various abbreviations for expressions in the formal syntax, which ultimately have no such parentheses.

We can formulate the definition of terms and formulas as a simultaneous recursion:

- An expression consisting of a single variable is a term. These will be the only terms consisting of a single symbol (unless we allow arity 0).
- Given a functional symbol $\mathbf{f}$ of arity n and given terms $\mathbf{A}_1, \ldots \mathbf{A}_n$, the expression $\mathbf{f}\mathbf{A}_1 \ldots \mathbf{A}_n$ is a term.
- Given a relational symbol $\mathbf{R}$ of arity n and given terms $\mathbf{A}_1, \ldots \mathbf{A}_n$, the expression $\mathbf{R}\mathbf{A}_1 \ldots \mathbf{A}_n$ is a formula.
- If $\mathbf{A}$ is a formula, then so is $\neg\mathbf{A}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are formulas, then so is $\vee\mathbf{A}\mathbf{B}$.
- If $\mathbf{A}$ is a formula and $\mathbf{x}$ is a variable, then $\tau_{\mathbf{x}}(\mathbf{A})$ is a term.

A *formative construction* is a finite sequence of terms and formulas where each expression in the list is the result of applying the rules specified above to previous terms of the list. It is purely a mechanical matter to check if a formative construction is correct.[7]

[7] Bourbaki's notion of formative construction is a bit different than this using concepts of "first species" and "second species" to identify potential terms and formulas. In fact, Bourbaki uses formative constructions to actually define terms and formulas instead of the above recursion. The approach is a bit different, but in the end the results are the same.

The Polish notation for $\vee$ is almost immediately abandoned in favor of the infix abbreviation:

Definition 3. We usually adopt infix notation for $\vee$, where $\mathbf{A} \vee \mathbf{B}$ is an informal way of indicating $\vee\mathbf{AB}$ when $\mathbf{A}$ and $\mathbf{B}$ are formulas. We also write "$\mathbf{A}$ or $\mathbf{B}$" or common English equivalents for $\vee\mathbf{AB}$. Likewise we write "not $\mathbf{A}$" or common English equivalents for $\neg\mathbf{A}$. Parenthesis are used to remove ambiguity from compounded expressions of this type.

Remark. Already from the definition we can formulate some meta-results about terms and formulas. For example, we can see from this description (reasoning in the metatheory) that every link in a term or formula involves an occurrence of τ with a following occurrence of $\square$. We can also see that every occurrence of $\square$ is linked to one (and only one) occurrence of τ.

We can see that formulas must start with $\neg$, $\vee$, or relational symbols. We can see that terms must either be a variable, or start with a functional symbol or τ. Given that a symbol cannot be both relational or functional, we see that the collection of terms is disjoint from the collection of expressions.

3.4 Axioms and theorems

Every theory can designate certain formulas as *simple axioms* (Bourbaki "explicit axioms"). Every variable occurring in a simple axiom is considered to be a *constant.* A theory can also have *axiom schemes.* An axiom scheme is a rule that assigns a formula to inputs. The inputs of an axiom scheme consists of zero or more terms and zero or more formulas. An axiom scheme must have the following closure property: the results of an axiom scheme are closed under the $(\mathbf{T}|\mathbf{x})$ operation for all terms $\mathbf{T}$ and variables $\mathbf{x}$. The simple axioms of the theorem and the outputs of the axiom schemes of a theory are considered *axioms* of the theory. The theorems of a theory are defined as follows:

- Every axiom is a theorem.
- If $\mathbf{A}$ and $\mathbf{B}$ are formulas and if $\vee\neg\mathbf{AB}$ and $\mathbf{A}$ are theorems, then $\mathbf{B}$ is a theorem.

This second rule is known as *modus ponens.* It is the only *rule of inference* in Bourbaki's system.

Definition 4. We use $\implies$ as an abbreviation for the expression $\vee\neg$. Thus if $\mathbf{A}$ and $\mathbf{B}$ are formulas then $\implies\mathbf{AB}$ is short for $\vee\neg\mathbf{AB}$. We usually employ infix notation and write $(\mathbf{A} \implies \mathbf{B})$ for $\implies\mathbf{AB}$. We write "if $\mathbf{A}$ then $\mathbf{B}$" or other common English equivalents for $\mathbf{A} \implies \mathbf{B}$.

So we can restate modus ponens as follows:[8]

[8]Commentary: Given the centrality of modus ponens to Bourbaki's system, it would have been preferable to make $\implies$ a primitive symbol. The extra reductionism makes the theory less elegant overall. Similarly using Polish notation, and the immediate avoidance of Polish notation is a bit strange. This is just an opinion, and the balance between clever reductionism and clarity can be a delicate one.

- if ($\mathbf{A} \implies \mathbf{B}$) and $\mathbf{A}$ are theorems, where $\mathbf{A}$ and $\mathbf{B}$ are formulas, then $\mathbf{B}$ is a theorem.

A proof can be regarded as a list of theorems where each theorem on the list is ether an axiom or the result of applying modus ponens to two previous theorems on the list. We can also require, as part of the proof, an auxiliary formative construction which includes each formula in the proof and each formula or term used as input to the axiom scheme. If we set this up right, checking the validity of a proof is a purely mechanical task (assuming the axiom schemes are implemented in a mechanical manner).

All axioms required in the theory of sets are listed by Bourbaki on page 414.

3.5 Mathematical theories

For a theory to be formalized in this language we need to list the specific relational and functional symbols in our theory. We also need to assign such symbols their arities (with a default value of 2). We also need to list the axioms and axiom schemes. Once we have done so we formulated what Bourbaki calls a *mathematical theory* or a *theory* for short (a better term might be a *Bourbaki type formal theory*). The above definitions then define terms, formulas, and theorems for the theory.

3.6 Semantics

I mentioned a few of the semantical ideals that motivate the above formal language, but only briefly. Here is a more detailed treatment. Since here the semantics is motivational, we can be less rigorous in our descriptions and arguments concerning semantics: once the formal system is set up, these semantical ideas do not affect whether or not a given expression is a formula, or whether or not a given formula is a theorem.

A theory is based on the idea that we postulate the existence of a nonempty collection of objects for which we want to prove theorems. We call such a collection a "universe". An interpretation for a collection of variables is a choice of object in this universe for each variable. Every term has the property that it designates an object of the universe for every interpretation of its variables. Every formula has the the property that it is either true or false for every interpretation of its variables.

A simple axiom is a formula that is supposed to be true for some interpretation of the variables occurring in the axioms. (Variables occurring in simple axioms are thought of as having a fixed interpretation). The interpretation of the variables for the simple axioms are thought of as being fixed throughout. Axiom schemes are supposed to produce formulas that are true regardless of the interpretations of its variables. A theorem is a formula that is true for all interpretations of the non-constant variables (and for the given interpretation of the constants).

Suppose $\mathbf{f}$ is a functional symbol of arity n. Then there should be a rule associated with $\mathbf{f}$ that assigns to each n-tuple of elements of the universe an element of the universe. Given an interpretation of the variables in $\mathbf{f}\mathbf{A}_1 \ldots \mathbf{A}_n$, suppose c_i is the object of the universe associated with $\mathbf{A}_i$, then $\mathbf{f}\mathbf{A}_1 \ldots \mathbf{A}_n$ is interpreted as the results of applying the rule for $\mathbf{f}$ to $(c_1, \ldots, c_n)$.

Suppose $\mathbf{R}$ is a relational symbol of arity n. Then there should be a rule associated with $\mathbf{R}$ that assigns to each n-tuple of elements of the universe a truth value (true or false). For any interpretation of the variables in $\mathbf{R}\mathbf{A}_1 \ldots \mathbf{A}_n$, suppose c_i is the object of the universe associated with $\mathbf{A}_i$, then $\mathbf{R}\mathbf{A}_1 \ldots \mathbf{A}_n$ is interpreted as the results of applying the rule for $\mathbf{R}$ to $(c_1, \ldots, c_n)$.

Suppose $\mathbf{A}$ is a formula. For each interpretation of its variables, $\neg\mathbf{A}$ is assigned the opposite truth value as $\mathbf{A}$. Suppose $\mathbf{A}$ and $\mathbf{B}$ are formulas. For each interpretation of the variables of $\vee\mathbf{AB}$, we assign this formula the truth value of true if and only if at least one of $\mathbf{A}$ and $\mathbf{B}$ is assigned true.

The semantics for τ and $\Box$ are a bit more subtle. We can think of τ as a sort of "description-choice" operator where $\tau_{\mathbf{x}}(\mathbf{A})$ is a choice, if possible, of an object satisfying the description given by A. It is easiest to make this more precise if we assume that the universe has a designated well-ordering. Consider a term of the form $\tau_{\mathbf{x}}(\mathbf{A})$, and note that $\mathbf{x}$ does not occur in $\tau_{\mathbf{x}}(\mathbf{A})$. Fix an interpretation of all the variables of $\tau_{\mathbf{x}}(\mathbf{A})$, but assume that $\mathbf{x}$ is uninterpreted for now. If there is an interpretation of $\mathbf{x}$ that makes $\mathbf{A}$ true (where the other variables are interpreted as before) then we choose the first object according to the well-ordering of the universe that makes $\mathbf{A}$ true. If no such object exists, just take the first object of the universe (we assume the universe is nonempty). Then $\tau_{\mathbf{x}}(\mathbf{A})$ is interpreted as that choice of object. Note that this semantics assumes the universe can be well-ordered, and it is not surprising that the axioms for Bourbaki's quantificational theory will yield the axiom of choice as a consequence.[9]

3.7 Metatheoretic results: expressions in general

We can prove various metatheorem about expressions. I will skip the proofs when they are straightforward (in a reasonable metatheory), and provide some hints when they are less straightforward.

Metatheorem 1. *The variable* $\mathbf{x}$ *does not occur in* $\tau_{\mathbf{x}}(\mathbf{A})$. *If* $\mathbf{y}$ *is not the same as* $\mathbf{x}$, *then* $\mathbf{y}$ *occurs in* $\tau_{\mathbf{x}}(\mathbf{A})$ *if and only if* $\mathbf{y}$ *occurs in* $\mathbf{A}$.

Metatheorem 2. *The expression* $(\mathbf{B}|\mathbf{x})\mathbf{A}$ *is* $\mathbf{A}$ *if there is no occurrence of* $\mathbf{x}$ *in the expression* $\mathbf{A}$. *In particular,* $(\mathbf{B}|\mathbf{x})\tau_{\mathbf{x}}(\mathbf{A})$ *is identical with* $\tau_{\mathbf{x}}(\mathbf{A})$.

Metatheorem 3. *Suppose* $\mathbf{C}$ *is an expression and* $\mathbf{x}$ *is a variable. Then the operator* $(\mathbf{C}|\mathbf{x})$ *commutes with juxtaposition. In other words, if* $\mathbf{A}$ *and* $\mathbf{B}$ *are expressions, then*

$$(\mathbf{C}|\mathbf{x})(\mathbf{AB}) \text{ is the same as } ((\mathbf{C}|\mathbf{x})\mathbf{A})((\mathbf{C}|\mathbf{x})\mathbf{B}).$$

Metatheorem 3 is fairly self-evident, but we do need to be cautious about links. Recall links cannot connect a variable $\mathbf{x}$ to other symbols which simplifies our metalinguistic reasoning. We have the following generalization of Metatheorem 3 that takes into account links (it is a technical result not found in Bourbaki, but I found it useful to help justify some of the following results):

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[9]Bourbaki does not mention that the semantics requires a well-ordered universe; the well-ordering idea is my interpretation. They say that $\tau_{\mathbf{x}}(\mathbf{A})$ is thought of as denoting some "distinguished object" with the desired property, if it exist, but otherwise "represents an object about which nothing can be said".

Metatheorem 4. *Suppose* $\mathbf{A}$ *is an expression that is formed by adding links to a juxtaposition* $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n$ *of expressions. Suppose that for each such added link there is a pair* (i,j) *of distinct integers such that* $\mathbf{A}_i$ *and* $\mathbf{A}_j$ *are nonvariable symbols (i.e., expressions consisting of a single symbol which is not a variable), and that the link in question connects* $\mathbf{A}_i$ *to* $\mathbf{A}_j$*.*

Then, for each expression $\mathbf{B}$ *and each variable* $\mathbf{x}$*, the expression* $(\mathbf{B}|\mathbf{x})\mathbf{A}$ *is the expression obtained by adding links to* $\mathbf{A}'_1\mathbf{A}'_2\cdots\mathbf{A}'_n$ *where* $\mathbf{A}'_i$ *is* $(\mathbf{B}|\mathbf{x})\mathbf{A}_i$ *(and so where* $\mathbf{A}'_i$ *is just* $\mathbf{A}_i$ *when* $\mathbf{A}_i$ *is a single nonvariable symbol); in particular, we add a link connected* $\mathbf{A}'_i$ *and* $\mathbf{A}'_j$ *for every added link of* $\mathbf{A}$ *connecting* $\mathbf{A}_i$ *and* $\mathbf{A}_j$*.*

(Note: some of the individual expressions $\mathbf{A}_i$*, and hence* $\mathbf{A}'_i$*, can have "internal links" not specified here.)*

Bourbaki uses "CS" to label various metatheoretic "criteria of substitution".

Metatheorem 5 (Bourbaki **CS1**)**.** *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are expressions and that* $\mathbf{x}$ *and* $\mathbf{y}$ *are variables. If* $\mathbf{y}$ *does not appear in* $\mathbf{A}$ *or if* $\mathbf{y}$ *is equal to* $\mathbf{x}$ *then*

$$(\mathbf{B}|\mathbf{x})\mathbf{A} \ \text{ is the same as } \ (\mathbf{B}|\mathbf{y})(\mathbf{y}|\mathbf{x})\mathbf{A}.$$

Proof. This is not too hard to see directly. Another path is to use Metatheorem 4 to reduce to the case where $\mathbf{A}$ consists of a single variable. (Note also that the case where $\mathbf{x}$ and $\mathbf{y}$ are the same is clear, so we can assume they are distinct in the proof). □

The following will be used in the theory of quantifiers:

Metatheorem 6 (Bourbaki **CS2**)**.** *Suppose that* $\mathbf{A},\mathbf{B}$ *and* $\mathbf{C}$ *are expressions and that* $\mathbf{x}$ *and* $\mathbf{y}$ *are distinct variables. If* $\mathbf{x}$ *does not appear in* $\mathbf{B}$ *then*

$$(\mathbf{B}|\mathbf{y})(\mathbf{C}|\mathbf{x})\mathbf{A} \ \text{ is the same as } \ (\mathbf{C}'|\mathbf{x})(\mathbf{B}|\mathbf{y})\mathbf{A}$$

where $\mathbf{C}'$ *is* $(\mathbf{B}|\mathbf{y})\mathbf{C}$*.*

Proof. We use Metatheorem 4 to reduce to the case where $\mathbf{A}$ consists of a single variable. □

Metatheorem 7 (Bourbaki **CS3**)**.** *Suppose that* $\mathbf{A}$ *is an expression and that* $\mathbf{x}$ *and* $\mathbf{y}$ *are variables. If* $\mathbf{y}$ *does not appear in* $\mathbf{A}$ *or if* $\mathbf{y}$ *is equal to* $\mathbf{x}$ *then*

$$\tau_{\mathbf{x}}\mathbf{A} \ \text{ is the same as } \ \tau_{\mathbf{y}}\left((\mathbf{y}|\mathbf{x})\mathbf{A}\right).$$

Metatheorem 8 (Bourbaki **CS4**)**.** *Suppose that* $\mathbf{B}$ *is an expression and that* $\mathbf{x}$ *and* $\mathbf{y}$ *are variables. Suppose that* $\mathbf{x}$ *is distinct from* $\mathbf{y}$*, and that* $\mathbf{x}$ *does not occur in* $\mathbf{B}$*. Then* $(\mathbf{B}|\mathbf{y})$ *and* $\tau_{\mathbf{x}}$ *are commuting operators. In other words, for all expressions* $\mathbf{A}$*,*

$$(\mathbf{B}|\mathbf{y})\tau_{\mathbf{x}}\mathbf{A} \ \text{ is the same as } \ \tau_{\mathbf{x}}\left((\mathbf{B}|\mathbf{y})\mathbf{A}\right).$$

Metatheorem 9 (Bourbaki **CS5**)**.** *Suppose that* $\mathbf{A},\mathbf{B}$ *and* $\mathbf{C}$ *are expressions and that* $\mathbf{x}$ *is a variable. Then*

$$(\mathbf{C}|\mathbf{x})(\neg\mathbf{A}) \text{ is the same as } \neg\left((\mathbf{C}|\mathbf{x})\mathbf{A}\right)$$

and

$$(\mathbf{C}|\mathbf{x})(\vee\mathbf{AB}) \textit{ is the same as } \vee((\mathbf{C}|\mathbf{x})\mathbf{A})((\mathbf{C}|\mathbf{x})\mathbf{B}).$$

In particular

$$(\mathbf{C}|\mathbf{x})(\mathbf{A} \implies \mathbf{B}) \textit{ is the same as } ((\mathbf{C}|\mathbf{x})\mathbf{A}) \implies ((\mathbf{C}|\mathbf{x})\mathbf{B}).$$

Proof. This follows from Metatheorem 3. □

3.8 Metatheoretic results: terms and expressions

We have various metatheorem about terms and formulas. I mentioned some in the remarks at the end of Section 3.3, and restate them here for convenient reference.

Metatheorem 10. *If an expression is a term then it is a single variable, or it starts with a τ or a functional symbol. If an expression is a relation, it begins with a relational symbol, $\neg$, or $\vee$. Thus terms and relations are disjoint syntactic categories.*

Also from the definition for term and formula we get the following (Bourbaki uses "CF" to label various metatheoretic "formative criteria"):

Metatheorem 11 (Bourbaki **CF1**)**.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas in a given theory then so is* $\vee\mathbf{AB}$*. (We will commonly abbreviate this as* $\mathbf{A} \vee \mathbf{B}$*).*

Metatheorem 12 (Bourbaki **CF2**)**.** *If* $\mathbf{A}$ *is a formula in a given theory then so is* $\neg\mathbf{A}$*.*

Metatheorem 13 (Bourbaki **CF3**)**.** *If* $\mathbf{A}$ *is a formula in a given theory and if* $\mathbf{x}$ *is a variable, then* $\tau_{\mathbf{x}}\mathbf{A}$ *is a term of the given theory. In other words, we can view* $\tau_{\mathbf{x}}$ *as an operator taking formulas to terms.*

Metatheorem 14 (Bourbaki **CF4**)**.** *If* $\mathbf{R}$ *is a relational symbol of arity n in a given theory and if* $\mathbf{A}_1, \ldots \mathbf{A}_n$ *are n terms in the given theory, then* $\mathbf{RA}_1 \cdots \mathbf{A}_n$ *is a formula of the given theory.*

If $\mathbf{f}$ *is a functional symbol of arity n in a given theory and if* $\mathbf{A}_1, \ldots \mathbf{A}_n$ *are n terms in the given theory, then* $\mathbf{fA}_1 \cdots \mathbf{A}_n$ *is a term of the given theory.*

Metatheorem 15 (Bourbaki **CF5**)**.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas in a given theory then so is* $\implies \mathbf{AB}$ *(which we commonly write as* $\mathbf{A} \implies \mathbf{B}$*)*

Bourbaki has a **CF6**, which is not really essential to the approach taken here, and **CF7** is a simple consequence of **CF8**. So we will jump to Bourbaki's **CF8**, which is essential to using the language.

Metatheorem 16 (Bourbaki **CF8**)**.** *Let* $\mathbf{T}$ *be a term of the given theory and let* $\mathbf{x}$ *be a variable. If* $\mathbf{A}$ *is a term of a given theory, then so is* $(\mathbf{T}|\mathbf{x})\mathbf{A}$*. If* $\mathbf{A}$ *is a formula of a given theory, then so is* $(\mathbf{T}|\mathbf{x})\mathbf{A}$*. In other words, we can view* $(\mathbf{T}|\mathbf{x})$ *as an operator that takes terms to terms and formulas to formulas.*

Proof. Use (strong) induction on the number of symbols of the formula: we consider formulas and terms with n symbols, and prove that the transformation $(\mathbf{T}|\mathbf{x})$ sends such formulas to formulas and such terms to terms. (The resulting formula and term are allowed to have an arbitrary number of symbols depending on $\mathbf{T}$).

We start with $n = 1$. For terms consisting of a single variable the result is clear. Other possible cases of length one (which we worry about only for theories allowing symbols of arity zero) are immediate since $(\mathbf{T}|\mathbf{x})$ leaves such expressions unchanged.

Now we consider the case where $n > 1$ using strong induction. So assume $\mathbf{A}$ a term or formula with n symbols. If the first symbol of $\mathbf{A}$ is not τ, then just use Metatheorem 3 and the induction hypothesis to conclude $(\mathbf{T}|\mathbf{x})\mathbf{A}$ is a term or formula (depending of $\mathbf{A}$).

The remaining case is where $\mathbf{A}$ is $\tau_{\mathbf{y}}\mathbf{B}$ for some formula $\mathbf{B}$ and variable $\mathbf{y}$. Let $\mathbf{u}$ be a variable distinct from $\mathbf{x}$ and $\mathbf{y}$ and not occurring in $\mathbf{B}$ or $\mathbf{T}$. We use CS3 (Metatheorem 7) to write $\tau_{\mathbf{y}}\mathbf{B}$ as $\tau_{\mathbf{u}}(\mathbf{u}|\mathbf{y})\mathbf{B}$. By CS4 (Metatheorem 8), the operators $(\mathbf{T}|\mathbf{x})$ and $\tau_{\mathbf{u}}$ commute. Thus $(\mathbf{T}|\mathbf{x})\tau_{\mathbf{y}}\mathbf{B}$ is $(\mathbf{T}|\mathbf{x})\tau_{\mathbf{u}}(\mathbf{u}|\mathbf{y})\mathbf{B}$, which in turn is in turn is just $\tau_{\mathbf{u}}(\mathbf{T}|\mathbf{x})(\mathbf{u}|\mathbf{y})\mathbf{B}$. By the induction hypothesis $(\mathbf{u}|\mathbf{y})\mathbf{B}$ is a formula, and clearly it has the same number of symbols as $\mathbf{B}$. By the induction hypothesis again, $(\mathbf{T}|\mathbf{x})(\mathbf{u}|\mathbf{y})\mathbf{B}$ is a formula. We conclude that $\mathbf{A}$, which is $\tau_{\mathbf{u}}(\mathbf{T}|\mathbf{x})(\mathbf{u}|\mathbf{y})\mathbf{B}$, is a term (by CF3, Metatheorem 13). □

3.9 Metatheoretic results: theorems

There are a few basic results called "deductive criteria" by Bourbaki which shorten proofs in the sense that they allow us to more easily establish a given formula as a theorem. These rules are designated with the letter 'C' by Bourbaki.

We start by restating modus ponens (also called *Syllogism* by Bourbaki, or the elimination rule for $\implies$ by some other authors):

Metatheorem 17 (Bourbaki's **C1**). *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas of a theory. If* $\mathbf{A} \implies \mathbf{B}$ *and* $\mathbf{A}$ *are theorems then so is* $\mathbf{B}$.

Definition 5. If $\mathcal{T}$ is a theory, if $\mathbf{T}$ is a term of $\mathcal{T}$, and if $\mathbf{x}$ is a variable then $(\mathbf{T}|\mathbf{x})\mathcal{T}$ denotes a theory with the same symbols (with the same classifications and arities) and axiom schemes as $\mathcal{T}$ but whose simple axioms are those formulas of the for $(\mathbf{T}|\mathbf{x})\mathbf{A}$ where $\mathbf{A}$ is a simple axiom of $\mathcal{T}$.

Metatheorem 18 (Replacement of constants, Bourbaki's **C2**). *Suppose* $\mathcal{T}$ *is a theory,* $\mathbf{T}$ *is a term of* $\mathcal{T}$*, and* $\mathbf{x}$ *is a variable. If* $\mathbf{A}$ *is a theorem of* $\mathcal{T}$ *then* $(\mathbf{T}|\mathbf{x})\mathbf{A}$ *is a theorem of the theory* $(\mathbf{T}|\mathbf{x})\mathcal{T}$.

Proof. Write down a list of formulas $\mathbf{R}_1, \ldots, \mathbf{R}_n$ such that $\mathbf{R}_n$ is $\mathbf{A}$ and such that each formula is an axiom of $\mathcal{T}$ or is obtained by applying modus ponens to earlier formulas on the list. Now apply the operator $(\mathbf{T}|\mathbf{x})$ to all the formulas on the list. Use CS5 (Metatheorem 9) and the fact that the results of an axiom scheme are closed under the $(\mathbf{T}|\mathbf{x})$ operator. □

Metatheorem 19 (Bourbaki's **C3**). *Suppose* $\mathcal{T}$ *is a theory and that* $\mathbf{T}$ *is a term of* $\mathcal{T}$*. Suppose* $\mathbf{x}$ *is a variable that is not a constant (does not occur in any simple axiom). If* $\mathbf{A}$ *is a theorem of* $\mathcal{T}$ *then* $(\mathbf{T}|\mathbf{x})\mathbf{A}$ *is a theorem of* $\mathcal{T}$.

Proof. This is a corollary of the previous result. □

Definition 6 (Stronger theory)**.** Suppose $\mathcal{T}$ and $\mathcal{T}'$ are theories. We say that $\mathcal{T}'$ is *stronger* than $\mathcal{T}$ if (1) every symbol of $\mathcal{T}$ is a symbol of $\mathcal{T}'$ with the same classifications and arity, (2) every axiom scheme of $\mathcal{T}$ is an axiom scheme of $\mathcal{T}'$, and (3) every simple axiom of $\mathcal{T}$ is a theorem of $\mathcal{T}'$

Metatheorem 20 (Bourbaki's **C4**)**.** *Suppose $\mathcal{T}'$ is a stronger theory than $\mathcal{T}$. Then every theorem of $\mathcal{T}$ is a theorem of $\mathcal{T}'$.*

Proof. Observe that every axioms of $\mathcal{T}$ is a theorem of $\mathcal{T}'$. If $\mathbf{A} \implies \mathbf{B}$ and $\mathbf{A}$ are theorems of $\mathcal{T}$ that are also theorems of $\mathcal{T}'$, where $\mathbf{A}$ and $\mathbf{B}$ are formulas of $\mathcal{T}$ and hence $\mathcal{T}'$, then $\mathbf{B}$ is a theorem of $\mathcal{T}'$ by modus ponens. □

Definition 7 (Equivalent theory)**.** Suppose $\mathcal{T}$ and $\mathcal{T}'$ are theories. We say that $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* if (1) they have the same symbols with the same classifications and arity (so have the same terms and formulas), (2) they have the same axiom schemes, and (3) every simple axiom of one is a theorem of the other.

In other words, a theories are equivalent if and only if each is stronger than the other.

Metatheorem 21 (Corollary of **C4**)**.** *Equivalent theories have the same terms, formulas, and theorems.*

Exercise 1 (Bourbaki's **C5**)**.** Suppose $\mathcal{T}$ is a theory, $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are terms of $\mathcal{T}$, and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are constants of $\mathcal{T}$. Suppose $\mathcal{T}'$ is a theory such that (1) every symbol of $\mathcal{T}$ is a symbol of $\mathcal{T}'$ (with the same classification and arity), (2) every axiom scheme of $\mathcal{T}$ is an axiom scheme of $\mathcal{T}'$, and (3) if $\mathbf{A}$ is a simple axiom of $\mathcal{T}$ then $(\mathbf{T}_1|\mathbf{a}_1)\cdots(\mathbf{T}_n|\mathbf{a}_n)\mathbf{A}$ is a theorem of $\mathcal{T}'$. Show that if $\mathbf{A}$ is a theorem of $\mathcal{T}$ then

$$(\mathbf{T}_1|\mathbf{a}_1)\cdots(\mathbf{T}_n|\mathbf{a}_n)\mathbf{A}$$

is a theorem of $\mathcal{T}'$.

Remark. I label this as an exercise since I do not know if this result is ever used by Bourbaki except as a way to explain the axiomatic method.

4 Propositional logic (§3 in Bourbaki)

Bourbaki does not use the term "propositional logic" but uses the term "logical theories" instead.

4.1 Axioms

Throughout Section 4 we assume we are in a theory $\mathcal{T}$ with at least the following axiom schemes:

S1. $(\mathbf{A} \vee \mathbf{A}) \implies \mathbf{A}$ is an axiom for any formula $\mathbf{A}$.

S2. $\mathbf{A} \implies (\mathbf{A} \vee \mathbf{B})$ is an axiom for any formulas $\mathbf{A}$ and $\mathbf{B}$.

S3. $(\mathbf{A} \vee \mathbf{B}) \implies (\mathbf{B} \vee \mathbf{A})$ is an axiom for any formulas $\mathbf{A}$ and $\mathbf{B}$.

S4. $(\mathbf{A} \implies \mathbf{B}) \implies ((\mathbf{C} \vee \mathbf{A}) \implies (\mathbf{C} \vee \mathbf{B}))$ is an axiom for any formulas $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$.

Metatheorem 22. *The above* S1, S2, S3, S4 *are in fact schemes: they are closed under application of the operator* $(\mathbf{T}|\mathbf{x})$ *where* $\mathbf{T}$ *is a term of the theory and* $\mathbf{x}$ *is a variable.*

4.2 Metatheorems for propositional logic

We mention **C1** (modus ponens) for reference:

Metatheorem 23 (Bourbaki's **C1**, modus ponens)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas of a theory. If* $\mathbf{A} \implies \mathbf{B}$ *and* $\mathbf{A}$ *are theorems then so is* $\mathbf{B}$.

Remark. Modus ponens can be considered to be the "$\implies$-elimination rule" since it converts a formula using $\implies$ to one where $\implies$ has been eliminated.

The definition of $\implies$ and S2 gives us the following:

Metatheorem 24. *Any theory with scheme S2 gives us theorems of the form*

$$\neg\mathbf{A} \implies (\mathbf{A} \implies \mathbf{B})$$

for all formulas $\mathbf{A}$ *and* $\mathbf{B}$ *of the theory.*

So for the next result we really only need $\mathcal{T}$ to satisfy S2 (of the four schemes listed above):

Metatheorem 25. *If* $\mathcal{T}$ *is contradictory in the sense that* $\mathbf{A}$ *and* $\neg\mathbf{A}$ *are theorems for some formula* $\mathbf{A}$*, then every formula is a theorem.*

Proof. Let $\mathbf{B}$ be any formula. We have $\neg\mathbf{A} \implies (\mathbf{A} \implies \mathbf{B})$ by the previous result. So $\mathbf{A} \implies \mathbf{B}$ is a theorem by modus ponens. Thus $\mathbf{B}$ is a theorem by modus ponens again. ☐

Metatheorem 26 (Bourbaki's **C6**)**.** *If* $\mathbf{A} \implies \mathbf{B}$ *and* $\mathbf{B} \implies \mathbf{C}$ *are theorems, then so is* $\mathbf{A} \implies \mathbf{C}$.

Proof. By S4 we have the axiom

$$(\mathbf{B} \implies \mathbf{C}) \implies \big((\mathbf{A} \implies \mathbf{B}) \implies (\mathbf{A} \implies \mathbf{C})\big).$$

Now use modus ponens twice. ☐

Metatheorem 27 (Bourbaki's **C7**)**.** *The formula* $\mathbf{B} \implies (\mathbf{A} \vee \mathbf{B})$ *is a theorem for any formulas* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. Use **C6** (Metatheorem 26) applied to S2 and S3. ☐

Metatheorem 28 (Introduction rules for $\vee$)**.** *If* $\mathbf{A}$ *is a theorem then so are* $\mathbf{A} \vee \mathbf{B}$ *and* $\mathbf{B} \vee \mathbf{A}$.

Proof. Use S2 and **C7** (Metatheorem 27) together with modus ponens. □

Metatheorem 29 (Bourbaki's **C8**)**.** *The formula* $\mathbf{A} \implies \mathbf{A}$ *is a theorem for any formula* $\mathbf{A}$*.*

Proof. Use **C6** (Metatheorem 26) applied to S2 and S1. □

Metatheorem 30 (Bourbaki's **C9**)**.** *Suppose* $\mathbf{B}$ *is a theorem. Then* $\mathbf{A} \implies \mathbf{B}$ *is a theorem for any formula* $\mathbf{A}$*.*

Proof. Use **C7** (Metatheorem 27) with $\mathbf{A}$ replaced by $\neg\mathbf{A}$. □

Metatheorem 31 (Bourbaki's **C10**, excluded middle)**.** *The formula* $\mathbf{A} \vee \neg\mathbf{A}$ *is a theorem for any formula* $\mathbf{A}$*.*

Proof. Use **C8** (Metatheorem 29) and the definition of $\implies$. Use S3 to switch order. □

Metatheorem 32 (Bourbaki's **C11**)**.** *The formula* $\mathbf{A} \implies \neg\neg\mathbf{A}$ *is a theorem for any formula* $\mathbf{A}$*.*

Proof. Apply **C10** (Metatheorem 31) to $\neg\mathbf{A}$, and use the definition of $\implies$. □

Metatheorem 33 (Bourbaki's **C12**, contrapositive)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas. Then*

$$(\mathbf{A} \implies \mathbf{B}) \implies \big((\neg\mathbf{B}) \implies (\neg\mathbf{A})\big)$$

is a theorem.

Proof. Combine **C11** (Metatheorem 32) with S4 and modus ponens to get

$$(\neg\mathbf{A}) \vee \mathbf{B} \implies (\neg\mathbf{A}) \vee (\neg\neg\mathbf{B}).$$

Use S3 and **C6** (Metatheorem 26) to switch the order in the last $\vee$. Now use the definition of $\implies$. □

Metatheorem 34 (Bourbaki's **C13**)**.** *Suppose* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are formulas and that* $\mathbf{A} \implies \mathbf{B}$ *is a theorem. Then*

$$(\mathbf{B} \implies \mathbf{C}) \implies (\mathbf{A} \implies \mathbf{C})$$

is a theorem.

Proof. Use the previous result (**C12**) to get the theorem $(\neg\mathbf{B}) \implies (\neg\mathbf{A})$. Use S4 to get $(\mathbf{C} \vee (\neg\mathbf{B})) \implies (\mathbf{C} \vee (\neg\mathbf{A}))$. Use S3 and **C6** (Metatheorem 26) (twice) to get the theorem

$$((\neg\mathbf{B}) \vee \mathbf{C}) \implies ((\neg\mathbf{A}) \vee \mathbf{C}).$$

Use the definition of $\implies$. □

Metatheorem 35 (Relative modus ponens)**.** *Suppose* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are formulas and that*

$$\mathbf{A} \implies (\mathbf{B} \implies \mathbf{C}) \qquad \textit{and} \qquad \mathbf{A} \implies \mathbf{B}$$

are theorems. Then $\mathbf{A} \implies \mathbf{C}$ *is a theorem.*

Proof. Use **C13** to get the theorem $(\mathbf{B} \implies \mathbf{C}) \implies (\mathbf{A} \implies \mathbf{C})$. Use **C6** (Metatheorem 26) to combine this with one of the given theorems to get

$$\mathbf{A} \implies (\mathbf{A} \implies \mathbf{C}).$$

Write this theorem as $(\neg\mathbf{A}) \vee (\mathbf{A} \implies \mathbf{C})$, and use S3 to get $(\mathbf{A} \implies \mathbf{C}) \vee (\neg\mathbf{A})$.

Now, by S2 we obtain the theorem $(\neg\mathbf{A}) \implies (\mathbf{A} \implies \mathbf{C})$. From this and S4 we get the theorem

$$((\mathbf{A} \implies \mathbf{C}) \vee (\neg\mathbf{A})) \implies ((\mathbf{A} \implies \mathbf{C}) \vee (\mathbf{A} \implies \mathbf{C})),$$

and so the theorem $(\mathbf{A} \implies \mathbf{C}) \vee (\mathbf{A} \implies \mathbf{C})$ (modus ponens). From this and S1 we get the theorem $\mathbf{A} \implies \mathbf{C}$. □

Metatheorem 36 (Bourbaki's **C14**, also called the "$\implies$-introduction rule", or "the method of auxiliary hypothesis")**.** *Let $\mathcal{T}'$ be the theory obtained by adding the formula $\mathbf{A}$ to the simple axioms of $\mathcal{T}$. If $\mathbf{B}$ is a theorem of $\mathcal{T}'$ then $\mathbf{A} \implies \mathbf{B}$ is a theorem of $\mathcal{T}$.*

Proof. Let $\mathbf{B}_1, \dots, \mathbf{B}_n$ be a sequence of formulas of $\mathcal{T}'$ ending in $\mathbf{B}$ such that each $\mathbf{B}_i$ is an axiom of $\mathcal{T}'$ or an application of modus ponens to two previous formulas in the sequence.

Suppose $\mathbf{B}_i$ is an axiom of $\mathcal{T}'$. There are two cases: either $\mathbf{B}_i$ is $\mathbf{A}$, or $\mathbf{B}_i$ is in fact an axiom of $\mathcal{T}'$. In the first case, $\mathbf{A} \implies \mathbf{B}_i$ is a theorem of $\mathcal{T}$ by **C8** (Metatheorem 29). In the second case, $\mathbf{A} \implies \mathbf{B}_i$ is a theorem of $\mathcal{T}$ by **C9** (Metatheorem 30). So $\mathbf{A} \implies \mathbf{B}_i$ is a theorem of $\mathcal{T}$ for any axiom $\mathbf{B}_i$ of $\mathcal{T}'$.

We prove by induction that $\mathbf{A} \implies \mathbf{B}_i$ is a theorem of $\mathcal{T}$ for each $\mathbf{B}_i$ in the sequence. If $\mathbf{B}_i$ is an axiom of $\mathcal{T}'$ we have the result already. So assume $\mathbf{B}_i$ is obtained by applying modus ponens to $\mathbf{B}_k$ and $\mathbf{B}_l$ where $k, l < i$ and where $\mathbf{B}_k$ is of the form $\mathbf{B}_l \implies \mathbf{B}_i$. By induction we can assume that $\mathbf{A} \implies \mathbf{B}_l$ and $\mathbf{A} \implies (\mathbf{B}_l \implies \mathbf{B}_i)$ are theorems of $\mathcal{T}$. So $\mathbf{A} \implies \mathbf{B}_i$ is a theorem of $\mathcal{T}$ by relative modus ponens (Metatheorem 35 above). □

Metatheorem 37 (Bourbaki's **C15**, method of *reductio ad absurdum*)**.** *Let $\mathcal{T}'$ be the theorem obtained by adding the formula $\neg\mathbf{A}$ as a simple axiom of $\mathcal{T}$. If $\mathcal{T}'$ is contradictory, then $\mathbf{A}$ is a theorem of $\mathcal{T}$.*

Proof. By Metatheorem 25 every formula is a theorem of $\mathcal{T}'$. In particular, $\mathbf{A}$ is a theorem of $\mathcal{T}'$. So $\neg\mathbf{A} \implies \mathbf{A}$ is a theorem of $\mathcal{T}$ by **C14** above. Using S4 we get the theorem $(\mathbf{A} \vee \neg\mathbf{A}) \implies (\mathbf{A} \vee \mathbf{A})$. By **C10** (excluded middle, Metatheorem 31) we get the theorem $\mathbf{A} \vee \mathbf{A}$. Now use S1. □

Metatheorem 38 (Bourbaki's **C16**)**.** *If $\mathbf{A}$ is a formula then $\neg\neg\mathbf{A} \implies \mathbf{A}$ is a theorem.*

Proof. Assume $\neg\neg\mathbf{A}$. Assume $\neg\mathbf{A}$. Then we have a contradiction. So (dropping assumption $\neg\mathbf{A}$ according to **C15** above) we have $\mathbf{A}$. Thus $(\neg\neg\mathbf{A}) \implies \mathbf{A}$ (dropping assumption $\neg\neg\mathbf{A}$ according to **C14** above). □

Metatheorem 39 (*reductio ad absurdum, or* $\neg$*-introduction* variant)**.** *Let* $\mathcal{T}'$ *be the theorem obtained by adding the formula* $\mathbf{A}$ *as a simple axiom of* $\mathcal{T}$*. If* $\mathcal{T}'$ *is contradictory, then* $\neg\mathbf{A}$ *is a theorem of* $\mathcal{T}$*.*

Proof. Since $\mathcal{T}'$ is contradictory, every formula is a theorem of $\mathcal{T}'$. Thus $\mathbf{A} \implies \neg\mathbf{A}$ is a theorem of $\mathcal{T}$. Now suppose $\neg\neg\mathbf{A}$. Under this supposition we have $\mathbf{A}$ (by **C16** above) and further we have $\neg\mathbf{A}$ by modus ponens. This is a contradiction. Thus $\neg\mathbf{A}$ by the method of *reductio ad absurdum* (**C15** above). □

Metatheorem 40 (Bourbaki's **C17**)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas. Then*

$$(\neg\mathbf{B} \implies \neg\mathbf{A}) \implies (\mathbf{A} \implies \mathbf{B})$$

is a theorem.

Proof. Assume $\neg\mathbf{B} \implies \neg\mathbf{A}$. Assume $\mathbf{A}$. Assume $\neg\mathbf{B}$. Then $\neg\mathbf{A}$ by modus ponens. This contradicts $\mathbf{A}$, so $\mathbf{B}$ is a theorem (reducto ad absurdum, dropping assumption $\neg\mathbf{B}$ according to **C15**). Thus $\mathbf{A} \implies \mathbf{B}$ is a theorem using only the first assumption (dropping assumption $\mathbf{A}$ according to **C14**). Using **C14** again gives the desired theorem. □

Metatheorem 41 (Bourbaki's **C18**, elimination rule for $\vee$, or the method of proof by cases)**.** *Suppose* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are formulas. If* $\mathbf{A} \vee \mathbf{B}$ *and* $\mathbf{A} \implies \mathbf{C}$ *and* $\mathbf{B} \implies \mathbf{C}$ *are theorems then so is* $\mathbf{C}$*.*

Proof. Chain together implications using S4 (twice) and S3 we have

$$\mathbf{A} \vee \mathbf{B} \implies \mathbf{C} \vee \mathbf{C}.$$

Now use modus ponens and S1. □

Corollary. *Suppose that* $\mathbf{A}$ *and* $\mathbf{C}$ *are formulas in a theory. If both* $\mathbf{A} \implies \mathbf{C}$ *and* $\neg\mathbf{A} \implies \mathbf{C}$ *are theorems, then so is* $\mathbf{C}$*.*

Proof. We have $\mathbf{A} \vee \neg\mathbf{A}$ by excluded middle (**C10**, metatheorem 31). Now use the above. □

The following will come into importance when we add quantifiers to our theory:

Metatheorem 42 (Bourbaki's **C19**, method of the auxiliary constant)**.** *Let* $\mathbf{x}$ *be a variable, let* $\mathbf{T}$ *be a term in a theory* $\mathcal{T}$*, and let* $\mathbf{A}, \mathbf{B}$ *be formulas in* $\mathcal{T}$*. Assume that* (1) $\mathbf{x}$ *not a constant (is not in any simple axioms) in* $\mathcal{T}$*,* (2) $\mathbf{x}$ *does not appear in* $\mathbf{B}$*,* (3) $(\mathbf{T}|\mathbf{x})\mathbf{A}$ *is a theorem of* $\mathcal{T}$*,* (4) $\mathbf{B}$ *is a theorem in* $\mathcal{T}'$ *where* $\mathcal{T}'$ *is the theory obtained by adding* $\mathbf{A}$ *as a simple axiom to* $\mathcal{T}$*. Then* $\mathbf{B}$ *is a theorem of* $\mathcal{T}$*.*

Proof. By **C14** (Metatheorem 36), $\mathbf{A} \implies \mathbf{B}$ is a theorem of $\mathcal{T}$. Since $\mathbf{x}$ is not a constant we get $(\mathbf{T}|\mathbf{x})(\mathbf{A} \implies \mathbf{B})$ as a theorem of $\mathcal{T}$ by **C3** (Metatheorem 19). Since $\mathbf{x}$ does not occur in $\mathbf{B}$, this theorem is just $((\mathbf{T}|\mathbf{x})\mathbf{A}) \implies \mathbf{B}$. Thus we get $\mathbf{B}$ as a theorem of $\mathcal{T}$ by modus ponens. □

Remark. Informally this explains the use of the phrase of the form "let x have the property P" in a proof where we view x as a temporary constant denoting a choice of object satisfying P. Suppose one can prove a result $\mathbf{B}$ with such an assumption, where $\mathbf{B}$ does not refer to the temporary constant x. Then if we know there is some object with property P (so a choice of x is possible) then we can conclude $\mathbf{B}$ independent of this assumption. This assumption acts merely as a convenience in the proof.

Actually a phrase of the form "let x have the property P", made as a temporary assumption, can signal two possible proof processes, and the reader needs to rely on context to tell which is being used. One possibility is that we prove a results $\mathbf{B}$ under such an assumption and conclude, by the method of auxiliary hypothesis (**C14**), that for any x satisfying P, we will have $\mathbf{B}$; here $\mathbf{B}$ usually depends on x. The second possibility is that we are trying to prove $\mathbf{B}$ independent of such an assumption. This second method (**C16**) requires that x not be mentioned in $\mathbf{B}$, and that something exists that satisfies P. In this case we often view x as a choice of objects satisfying P.

4.3 Conjunction

Next we consider conjunction:

Definition 8. Given formulas $\mathbf{A}$ and $\mathbf{B}$, we define $\mathbf{A} \wedge \mathbf{B}$ as an abbreviation for $\neg\big((\neg\mathbf{A}) \vee (\neg\mathbf{B})\big)$. We call $\mathbf{A} \wedge \mathbf{B}$ the *conjunction* of $\mathbf{A}$ and $\mathbf{B}$.

We have the associated formative criterion:

Metatheorem 43 (Bourbaki's **CF9**)**.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas then so is* $\mathbf{A} \wedge \mathbf{B}$*.*

We also have the associated criterion of substitution:

Metatheorem 44 (Bourbaki's **CS6**)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas,* $\mathbf{T}$ *is a term, and* $\mathbf{x}$ *is a variable. Then* $(\mathbf{T}|\mathbf{x})(\mathbf{A} \wedge \mathbf{B})$ *is the same as* $(\mathbf{T}|\mathbf{x})(\mathbf{A}) \wedge (\mathbf{T}|\mathbf{x})(\mathbf{B})$*.*

In practice the definition of conjunction is not used much in mathematical reasoning. The following introduction and elimination rules reflect better what is used in most mathematical reasoning.

Metatheorem 45 (Bourbaki's **C20**, introduction rule for $\wedge$)**.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are theorems then so is* $\mathbf{A} \wedge \mathbf{B}$*.*

Proof. Assume $\neg(\mathbf{A} \wedge \mathbf{B})$, which is $\neg\neg\big((\neg\mathbf{A}) \vee (\neg\mathbf{B})\big)$ by definition. Use **C16** (Metatheorem 38) to get the theorem $(\neg\mathbf{A}) \vee (\neg\mathbf{B})$. By definition of $\implies$, this is $\mathbf{A} \implies (\neg\mathbf{B})$. So we get $\neg\mathbf{B}$ (modus ponens). This gives a contradiction since we also have $\mathbf{B}$ as a theorem (by assumption). □

Metatheorem 46 (Bourbaki's **C21**, elimination rules for $\wedge$)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are formulas. Then* $\mathbf{A} \wedge \mathbf{B} \implies \mathbf{A}$ *and* $\mathbf{A} \wedge \mathbf{B} \implies \mathbf{B}$ *are theorems. In particular, if* $\mathbf{A} \wedge \mathbf{B}$ *is a theorem, then so are* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Proof. Assume $\mathbf{A} \wedge \mathbf{B}$, which is $\neg\big((\neg\mathbf{A}) \vee (\neg\mathbf{B})\big)$ by definition. We will show $\mathbf{A}$ is a theorem using **C15** (reductio ad absurdum), so assume $\neg\mathbf{A}$. So $(\neg\mathbf{A}) \vee (\neg\mathbf{B})$ by S2, a contradiction.

The proof of $\mathbf{A} \wedge \mathbf{B} \implies \mathbf{B}$ is similar except we use **C7** (Metatheorem 27) instead of S2. □

Definition 9. We define $\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_n$ and $\mathbf{A}_1 \vee \cdots \vee \mathbf{A}_n$ by grouping first on the right. For example, $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}$ is $\mathbf{A} \wedge \big(\mathbf{B} \wedge (\mathbf{C} \wedge \mathbf{D})\big)$.

Metatheorem 47. *Suppose $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are formulas. Then $\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_n$ is a theorem if and only if each $\mathbf{A}_i$ is a theorem.*

Remark. The above requires induction in the metalanguage (plus the introduction and elimination rules). You can avoid this use of induction by just proving the special cases actually needed.

4.4 Logical equivalence

Next we consider logical equivalence (Bourbaki: "equivalence"):

Definition 10. Given formulas $\mathbf{A}$ and $\mathbf{B}$, we define $\mathbf{A} \iff \mathbf{B}$ as an abbreviation for $(\mathbf{A} \implies \mathbf{B}) \wedge (\mathbf{B} \implies \mathbf{A})$.

We have the associated formative criterion:

Metatheorem 48 (Bourbaki's **CF10**)**.** *If $\mathbf{A}$ and $\mathbf{B}$ are formulas then the expression $\mathbf{A} \iff \mathbf{B}$ is a formula.*

We also have the associated criterion of substitution:

Metatheorem 49 (Bourbaki's **CS7**)**.** *Suppose $\mathbf{A}$ and $\mathbf{B}$ are formulas, $\mathbf{T}$ is a term, and $\mathbf{x}$ is a variable. Then $(\mathbf{T}|\mathbf{x})(\mathbf{A} \iff \mathbf{B})$ is the same as the formula $(\mathbf{T}|\mathbf{x})(\mathbf{A}) \iff (\mathbf{T}|\mathbf{x})(\mathbf{B})$.*

Bourbaki leaves the proofs of **C22** to **C25** to the reader using the proof techniques already established above.

Metatheorem 50 (Bourbaki's **C22**, plus reflexivity)**.** *Suppose $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are formulas. Then $\mathbf{A} \iff \mathbf{A}$ is a theorem. If $\mathbf{A} \iff \mathbf{B}$ is a theorem then so is $\mathbf{B} \iff \mathbf{A}$. If $\mathbf{A} \iff \mathbf{B}$ and $\mathbf{B} \iff \mathbf{C}$ are theorems then so is $\mathbf{A} \iff \mathbf{C}$.*

Metatheorem 51 (Bourbaki's **C23**)**.** *Suppose $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are formulas and that $\mathbf{A} \iff \mathbf{B}$ is a theorem. Then the following are theorems:*

- $(\neg\mathbf{A}) \iff (\neg\mathbf{B})$
- $(\mathbf{A} \implies \mathbf{C}) \iff (\mathbf{B} \implies \mathbf{C})$
- $(\mathbf{C} \implies \mathbf{A}) \iff (\mathbf{C} \implies \mathbf{B})$
- $(\mathbf{A} \wedge \mathbf{C}) \iff (\mathbf{B} \wedge \mathbf{C})$
- $(\mathbf{A} \vee \mathbf{C}) \iff (\mathbf{B} \vee \mathbf{C})$

Metatheorem 52 (Bourbaki's **C24**). *Suppose* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are formulas. Then the following are theorems:*

- $(\neg\neg\mathbf{A}) \iff \mathbf{A}$ *(combine **C11** and **C16**)*
- $(\mathbf{A} \implies \mathbf{B}) \iff ((\neg\mathbf{B}) \implies (\neg\mathbf{A}))$ *(combine **C12** and **C17**)*
- $(\mathbf{A} \wedge \mathbf{A}) \iff \mathbf{A}$
- $(\mathbf{A} \wedge \mathbf{B}) \iff (\mathbf{B} \wedge \mathbf{A})$
- $(\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})) \iff ((\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C})$
- $(\mathbf{A} \vee \mathbf{B}) \iff \neg((\neg\mathbf{A}) \wedge (\neg\mathbf{B}))$
- $(\mathbf{A} \vee \mathbf{A}) \iff \mathbf{A}$
- $(\mathbf{A} \vee \mathbf{B}) \iff (\mathbf{B} \vee \mathbf{A})$
- $(\mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})) \iff ((\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C})$
- $(\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C})) \iff ((\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C}))$
- $(\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C})) \iff ((\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C}))$
- $(\mathbf{A} \wedge (\neg\mathbf{B})) \iff \neg(\mathbf{A} \implies \mathbf{B})$
- $(\mathbf{A} \vee \mathbf{B}) \iff (\neg\mathbf{A} \implies \mathbf{B})$

Metatheorem 53 (Bourbaki's **C25**). *Suppose* $\mathbf{B}$ *is a formula. If* $\mathbf{A}$ *is a theorem then* $(\mathbf{A} \wedge \mathbf{B}) \iff \mathbf{B}$ *is a theorem. If* $\neg\mathbf{A}$ *is a theorem then* $(\mathbf{A} \vee \mathbf{B}) \iff \mathbf{B}$ *is a theorem.*

5 Quantifiers (§4.1 - 4.3 in Bourbaki)

Now we consider the logic that introduces quantifiers $\exists$ and $\forall$, as well as developing the τ quantifier. The logic has the axiom schemes S1, S2, S3, S4, and a new axiom scheme concerning τ:

S5. If $\mathbf{R}$ is a formula, $\mathbf{T}$ is a term, and $\mathbf{x}$ is a variable, then the following is an axiom:

$$(\mathbf{T}|\mathbf{x})\mathbf{R} \implies (\tau_{\mathbf{x}}(\mathbf{R})|\mathbf{x})\mathbf{R}.$$

Remark. Informally S5 can be interpreted as saying that if there is at least one object satisfying the property described by $\mathbf{R}$ then $\tau_{\mathbf{x}}(\mathbf{R})$ will be some choice of object satisfying the property.

We note that we will not need S5 until Section 5.4. First we define the quantifiers $\exists$ and $\forall$ in terms of τ and develop a few rules related to the definitions.

5.1 Bracket substitution notation

Bourbaki introduces a bracket substitution notation in their §1, but it is not really used it until their §5 (on theories with equality). We will accelerate the usage a bit by making use of it with quantifiers since it sometimes yields better notation.

Suppose $\mathbf{R}$ is a formula and $\mathbf{x}$ is a designated variable (the choice of variable is either mentioned explicity, or understood from context), then for each term $\mathbf{T}$ the formula $\mathbf{R}[\mathbf{T}]$ denotes the formula obtained by replacing all occurrences of $\mathbf{x}$ in $\mathbf{R}$ with $\mathbf{T}$. In other words, $\mathbf{R}[\mathbf{T}]$ is a notational variant for $(\mathbf{T}|\mathbf{x})\mathbf{R}$. In particular $\mathbf{R}[\mathbf{x}]$ is just $\mathbf{R}$. Actually Bourbaki does not require that $\mathbf{R}$ be a formula, it can be any expression, and we have a variant for more than one variable:[10]

Definition 11 (Substitution)**.** Let $\mathbf{x}$ be a variable. When we introduce an expression as $\mathbf{A}[\mathbf{x}]$ we are naming the expression $\mathbf{A}$, but we are also indicating that $\mathbf{x}$ is the designated variable. The variable $\mathbf{x}$ does not have to occur in $\mathbf{A}$, and $\mathbf{A}$ can have variables not equal to $\mathbf{x}$. If $\mathbf{T}$ is a term then $\mathbf{A}[\mathbf{T}]$ denotes the expression obtained by simultaneously replacing all occurrences of $\mathbf{x}$ in $\mathbf{A}$ with $\mathbf{T}$. Note: the variable $\mathbf{x}$ is allowed to appear in $\mathbf{T}$.

More generally, let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a list of distinct variables. When we introduce a term or formula as $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ we are naming it $\mathbf{A}$, but we are also indicating that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the designated list of variables. Note that $\mathbf{A}$ can have other variables not on this list, and not every variable on the list needs to actual occur in $\mathbf{A}$. If $\mathbf{T}_1, \ldots, \mathbf{T}_n$ is a list of n terms then $\mathbf{A}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$ will designate the expression obtained by simultaneously replacing all occurrences of $\mathbf{x}_i$ in $\mathbf{A}$ with $\mathbf{T}_i$. Note: the variable $\mathbf{x}_i$ is allowed to appear in the terms $\mathbf{T}_1, \ldots, \mathbf{T}_n$.

Metatheorem 54. *Suppose* $\mathbf{A}[\mathbf{x}]$ *is an expression and* $\mathbf{T}$ *is a term. Then* $\mathbf{A}[\mathbf{T}]$ *is equal to* $(\mathbf{T}|\mathbf{x})\mathbf{A}$*. In particular* (**CF8**, Metatheorem 16)*, if* $\mathbf{A}[\mathbf{x}]$ *is a term then so is* $\mathbf{A}[\mathbf{T}]$*, and if* $\mathbf{A}[\mathbf{x}]$ *is a formula then so is* $\mathbf{A}[\mathbf{T}]$.

Suppose $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ *is an expression and* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ *is a list of n terms, then*

$$\mathbf{A}[\mathbf{T}_1, \ldots, \mathbf{T}_n] \quad \textit{is} \quad (\mathbf{T}_1|\mathbf{u}_1)\cdots(\mathbf{T}_n|\mathbf{u}_n)(\mathbf{u}_1|\mathbf{x}_1)\cdots(\mathbf{u}_n|\mathbf{x}_n)\mathbf{A}$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_n$ *is a list of distinct variable chosen to be distinct from* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *and any variable in* $\mathbf{A}$ *or in any* $\mathbf{T}_i$*. In particular* (**CF8**)*, if* $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ *is a term then so is* $\mathbf{A}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$*, and if* $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ *is a formula then so is* $\mathbf{A}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$.

Here is a definition not in Bourbaki, but it is a useful concept for some syntactical considerations:

Definition 12 (Parameters)**.** Suppose $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is an expression with designated variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then any variable occurring in $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ that is not one of the designated variables is called a *parameter* of $\mathbf{A}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$.

5.2 Definitions of quantifiers

Now we introduce the *existential quantifier* $\exists$ and the *universal quantifier* $\forall$:

[10] Bourbaki sticks to one or two variables, which lightens the metamathematical prerequisites.

Definition 13. If $\mathbf{R}$ is a formula and if $\mathbf{x}$ is a variable, then $(\exists\mathbf{x})\mathbf{R}$ is defined to be an abbreviation for $(\tau_{\mathbf{x}}(\mathbf{R})|\mathbf{x})\mathbf{R}$. In other words, if $\mathbf{R}[\mathbf{x}]$ is a formula with designated variable $\mathbf{x}$ then $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$ is $\mathbf{R}[\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])]$.

Definition 14. If $\mathbf{R}$ is a formula and if $\mathbf{x}$ is a variable, then $(\forall\mathbf{x})\mathbf{R}$ is an abbreviation for $\neg((\exists\mathbf{x})\neg\mathbf{R})$.

Metatheorem 55 (Bourbaki's **CF11**)**.** *If* $\mathbf{R}$ *is a formula and if* $\mathbf{x}$ *is a variable then* $(\forall\mathbf{x})\mathbf{R}$ *and* $(\forall\mathbf{x})\mathbf{R}$ *are formulas, and the variable* $\mathbf{x}$ *does not occur in these formulas. Aside from* $\mathbf{x}$*, the formulas* $\mathbf{R}, (\exists\mathbf{x})\mathbf{R}$*, and* $(\forall\mathbf{x})\mathbf{R}$ *have the same variables.*

Proof. That we get formulas is a straightforward consequence of earlier results (such as **CF8**, Metatheorem 16). Recall that $\mathbf{x}$ does not occur in the term $\tau_{\mathbf{x}}(\mathbf{R})$ (Metatheorem 1). ☐

Remark. The fact that $\mathbf{x}$ does not occur in $(\forall\mathbf{x})\mathbf{R}[\mathbf{x}]$ is a bit surprising from the modern point of view. Today we might prefer saying that $\mathbf{x}$ does not occur *free* in $(\forall\mathbf{x})\mathbf{R}[\mathbf{x}]$, but does occur as a bound variable. However, Bourbaki's device of using τ linked with $\square$ in order to handle bound variables means that $\mathbf{x}$ does not occur at all in such formulas; where we might expect to find a $\mathbf{x}$ in the formula, we will find the box $\square$ instead.

5.3 Metatheorems for quantifiers: without S5

We have already introduced a few metatheorems for quantifiers above. The following allows us to change "dummy variables". Note that in Bourbaki's system changing bound variables does not just results in logically equivalent formulas, but actually the same formula:

Metatheorem 56 (Bourbaki's **CS8**)**.** *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Suppose that the variable* $\mathbf{y}$ *is not a parameter of* $\mathbf{R}[\mathbf{x}]$*. Then* $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is just* $(\exists\mathbf{y})\mathbf{R}[\mathbf{y}]$*. Similarly,* $(\forall\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is just* $(\forall\mathbf{y})\mathbf{R}[\mathbf{y}]$*.*

In the other substitution notation, we can formulate this as follows. If $\mathbf{R}$ *is a formula and if the variable* $\mathbf{y}$ *does not appear in* $\mathbf{R}$*, or if* $\mathbf{y}$ *is* $\mathbf{x}$*, then* $(\exists\mathbf{x})\mathbf{R}$ *is the same formula as* $(\exists\mathbf{y})(\mathbf{y}|\mathbf{x})\mathbf{R}$*, and* $(\forall\mathbf{x})\mathbf{R}$ *is the same formula as* $(\forall\mathbf{y})(\mathbf{y}|\mathbf{x})\mathbf{R}$*.*

Proof. Observe that $\mathbf{R}[\mathbf{y}]$ is just $(\mathbf{y}|\mathbf{x})\mathbf{R}[\mathbf{x}]$, and so

$$\tau_{\mathbf{y}}(\mathbf{R}[\mathbf{y}]) \quad \text{is} \quad \tau_{\mathbf{y}}((\mathbf{y}|\mathbf{x})\mathbf{R}[\mathbf{x}]) \quad \text{which is} \quad \tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])$$

by **CS3** (Metatheorem 7).

The formula $(\exists\mathbf{y})\mathbf{R}[\mathbf{y}]$ is $(\tau_{\mathbf{y}}(\mathbf{R}[\mathbf{y}]) \mid \mathbf{y})\,\mathbf{R}[\mathbf{y}]$, which by the above can be written as $(\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}]) \mid \mathbf{y})\,\mathbf{R}[\mathbf{y}]$. In other words, $(\exists\mathbf{y})\mathbf{R}[\mathbf{y}]$ is

$$(\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}]) \mid \mathbf{y})\,(\mathbf{y}|\mathbf{x})\;\mathbf{R}[\mathbf{x}],$$

which by **CS1** (Metatheorem 5) is $(\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}]) \mid \mathbf{x})\,\mathbf{R}[\mathbf{x}]$. By definition, this is just $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$.

The result for $\forall$ follows from the result for $\exists$ by definition of $\forall$. ☐

The next result allows us to move a substitution inside a quantifier as long as the substitution avoids the bound variable.

Metatheorem 57 (Bourbaki's **CS9**)**.** *Suppose that* $\mathbf{x}$ *and* $\mathbf{y}$ *are distinct variables. Suppose that* $\mathbf{R}[\mathbf{y}]$ *is a formula with designated variable* $\mathbf{y}$ *(where* $\mathbf{R}$ *can include the variable* $\mathbf{x}$*), and suppose* $\mathbf{U}$ *is a term such that* $\mathbf{x}$ *does not appear in* $\mathbf{U}$*. Then*

$$(\mathbf{U}|\mathbf{y})\Big((\exists\mathbf{x})\mathbf{R}[\mathbf{y}]\Big) \text{ is the same as } (\exists\mathbf{x})(\mathbf{R}[\mathbf{U}]).$$

and

$$(\mathbf{U}|\mathbf{y})\Big((\forall\mathbf{x})\mathbf{R}[\mathbf{y}]\Big) \text{ is the same as } (\forall\mathbf{x})(\mathbf{R}[\mathbf{U}]).$$

In particular, the operator $(\mathbf{U}|\mathbf{y})$ *commutes with both* $(\exists\mathbf{x})$ *and* $(\forall\mathbf{x})$*.*

Proof. Note that $(\mathbf{U}|\mathbf{y})\Big((\exists\mathbf{x})\mathbf{R}[\mathbf{y}]\Big)$ is $(\mathbf{U}|\mathbf{y})(\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{y}])|\mathbf{x})\mathbf{R}[\mathbf{y}]$ which is equal to $(\mathbf{T}|\mathbf{x})(\mathbf{U}|\mathbf{y})\mathbf{R}[\mathbf{y}]$, and so to $(\mathbf{T}|\mathbf{x})\mathbf{R}[\mathbf{U}]$, where $\mathbf{T}$ is $(\mathbf{U}|\mathbf{y})\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{y}])$ (see **CS2**, Metatheorem 6). By **CS4** (Metatheorem 8), $\mathbf{T}$ is just $\tau_{\mathbf{x}}((\mathbf{U}|\mathbf{y})\mathbf{R}[\mathbf{y}])$, which is $\tau_{\mathbf{x}}\mathbf{R}[\mathbf{U}]$.

So $(\mathbf{U}|\mathbf{y})\Big((\exists\mathbf{x})\mathbf{R}[\mathbf{y}]\Big)$ is $(\tau_{\mathbf{x}}\mathbf{R}[\mathbf{U}]|\mathbf{x})\mathbf{R}[\mathbf{U}]$, which by definition is $(\exists\mathbf{x})(\mathbf{R}[\mathbf{U}])$.

The result for universal quantifiers follows from that of existential quantifiers and the definition of universal quantifier. □

Metatheorem 58 (Bourbaki's **C19** updated for ∃; method of the auxiliary constant, or ∃-elimination rule)**.** *Let* $\mathbf{x}$ *be a variable, and let* $\mathbf{R}$ *and* $\mathbf{B}$ *be formulas. Assume that* (1) $\mathbf{x}$ *not a constant (is not in any simple axioms),* (2) $\mathbf{x}$ *does not appear in* $\mathbf{B}$*,* (3) $(\exists\mathbf{x})\mathbf{R}$ *is a theorem of* $\mathcal{T}$*,* (4) $\mathbf{B}$ *is a theorem in* $\mathcal{T}'$ *where* $\mathcal{T}'$ *is the theory obtained by adding* $\mathbf{R}$ *as a simple axiom to* $\mathcal{T}$*. Then* $\mathbf{B}$ *is a theorem of* $\mathcal{T}$*.*

Proof. Since $(\exists\mathbf{x})\mathbf{R}$ is a theorem, we have $(\mathbf{T}|\mathbf{x})\mathbf{R}$ as a theorem of $\mathcal{T}$ where $\mathbf{T}$ is $\tau_{\mathbf{x}}(\mathbf{R})$ (definition of existential quantifier). So we can use the original form of **C19** (Metatheorem 42) to conclude that $\mathbf{B}$ is a theorem of $\mathcal{T}$. □

We also have the following common variant:

Metatheorem 59 (∃-elimination rule)**.** *Let* $\mathbf{x}$ *be a variable and let* $\mathbf{R}[\mathbf{x}]$ *and* $\mathbf{B}$ *be formulas. Assume that* (1) $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is a theorem of* $\mathcal{T}$*,* (2) $\mathbf{c}$ *is an unused variable, or at least it is not a constant (is not in any simple axioms) and does not appear in* $\mathbf{B}$ *or* $\mathbf{R}[\mathbf{x}]$*,* (3) $\mathbf{B}$ *is a theorem in* $\mathcal{T}'$ *where* $\mathcal{T}'$ *is the theory obtained by adding* $\mathbf{R}[\mathbf{c}]$ *as a simple axiom to* $\mathcal{T}$*. Then* $\mathbf{B}$ *is a theorem of* $\mathcal{T}$*.*

Proof. This follows from the previous result together with the fact that $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$ is just $(\exists\mathbf{c})\mathbf{R}[\mathbf{c}]$ (see **CS8**, Metatheorem 56). □

Metatheorem 60 (*reductio ad absurdum* variant)**.** *Suppose* $(\exists\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is a theorem in a theory* $\mathcal{T}$ *where* $\mathbf{R}[\mathbf{x}]$ *is a formula of* $\mathcal{T}$ *with designated variable* $\mathbf{x}$*. Suppose the theory* $\mathcal{T}'$ *where we add* $\mathbf{R}[\mathbf{c}]$ *and* $\mathbf{A}$ *to* $\mathcal{T}$ *is contradictory, where* $\mathbf{c}$ *is a non-constant variable in* $\mathcal{T}$ *that does not appear in* $\mathbf{R}[\mathbf{x}]$ *or* $\mathbf{A}$*. Then* $\neg\mathbf{A}$ *is a theorem of* $\mathcal{T}$*.*

Proof. Since $\mathcal{T}'$ is contradictory, any formula, including $\neg\mathbf{A}$, is a theorem of $\mathcal{T}'$. In particular, $\mathbf{A} \implies \neg\mathbf{A}$ is a theorem of the theorem we get by adding $\mathbf{R}[\mathbf{c}]$ to $\mathcal{T}$. By $\exists$-introduction above, we have $\mathbf{A} \implies \neg\mathbf{A}$ as a theorem of $\mathcal{T}$. But this means $\neg\mathbf{A}$ is a theorem of $\mathcal{T}$. □

Metatheorem 61 (Bourbaki's **C26**)**.** *If* $\mathbf{R}[\mathbf{x}]$ *is a formula then*

$$(\forall\mathbf{x})\mathbf{R}[\mathbf{x}] \iff \mathbf{R}\big[\tau_{\mathbf{x}}(\neg\mathbf{R}[\mathbf{x}])\big]$$

is a theorem.

In terms of the other notation for substitution, this can result can be stated as follows. If $\mathbf{R}$ *is a formula and if* $\mathbf{x}$ *is a variable, then*

$$(\forall\mathbf{x})\mathbf{R} \iff \big(\tau_{\mathbf{x}}(\neg\mathbf{R}) \,|\, \mathbf{x}\big)\mathbf{R}$$

is a theorem.

Proof. Note that $(\forall\mathbf{x})\mathbf{R}[\mathbf{x}]$ is defined as $\neg(\exists\mathbf{x})(\neg\mathbf{R}[\mathbf{x}])$ which in turn is defined as $\neg\Big(\tau_{\mathbf{x}}(\neg\mathbf{R}[\mathbf{x}]) \,|\, \mathbf{x}\Big)(\neg\mathbf{R}[\mathbf{x}])$. This is $\neg\neg\mathbf{R}\big[\tau_{\mathbf{x}}(\neg\mathbf{R}[\mathbf{x}])\big]$. □

Metatheorem 62 ($\forall$-introduction rule, or Bourbaki's **C27**)**.** *Suppose* $\mathbf{R}$ *is a theorem and* $\mathbf{x}$ *is a variable that is not a constant (in the sense that it does not appear in any simple axiom). Then* $(\forall\mathbf{x})\mathbf{R}$ *is a theorem.*

Proof. By **C3** (Metatheorem 19) we get the theorem $\big(\tau_{\mathbf{x}}(\neg\mathbf{R}) \,|\, \mathbf{x}\big)\mathbf{R}$ from $\mathbf{R}$ since $\mathbf{x}$ is not a constant. By the above (**C26**) we get $(\forall\mathbf{x})\mathbf{R}$. □

Metatheorem 63 (Bourbaki's **C28**)**.** *If* $\mathbf{R}$ *is a formula then*

$$\neg(\forall\mathbf{x})\mathbf{R} \iff (\exists\mathbf{x})(\neg\mathbf{R})$$

is a theorem.

Proof. Start with the following which comes from the definition of $\forall$:

$$(\forall\mathbf{x})\mathbf{R} \iff \neg(\exists\mathbf{x})(\neg\mathbf{R}).$$

□

5.4 Metatheorems for quantifiers: using S5

Recall that S5 is the axiom scheme that produces

$$(\mathbf{T}|\mathbf{x})\mathbf{R} \implies \Big(\tau_{\mathbf{x}}(\mathbf{R})|\mathbf{x}\Big)\mathbf{R}$$

for each term $\mathbf{T}$, variable $\mathbf{x}$, and formula $\mathbf{R}$. So if $\mathbf{R}[\mathbf{x}]$ is a formula with designated variable $\mathbf{x}$ then S5 produces the axiom

$$\mathbf{R}[\mathbf{T}] \implies \mathbf{R}[\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])]$$

for each term $\mathbf{T}$. By definition of $\exists$, we can rephrase S5 as follows:

S5. If $\mathbf{T}$ is a term and $\mathbf{R}[\mathbf{x}]$ is a formula with designated variable $\mathbf{x}$, then the following is an axiom:

$$\mathbf{R}[\mathbf{T}] \implies (\exists \mathbf{x})\mathbf{R}[\mathbf{x}].$$

This formulation of S5 gives what is sometimes called "∃-introduction". In what follows we will assume S1 to S5, but first we must check that S5 is a well-behaved axiom scheme:

Metatheorem 64. *The rule* S5 *is an axiom scheme.*

Proof. Let $\mathbf{y}$ be a variable and let $\mathbf{U}$ be a term. We must show that if $\mathbf{A}$ is an axiom produced by S5, then so is $(\mathbf{U}|\mathbf{y})\mathbf{A}$. We will write $\mathbf{A}$ according to the original form of S5:

$$(\mathbf{T}|\mathbf{x})\mathbf{R} \implies \Big(\tau_{\mathbf{x}}(\mathbf{R}) \,|\, \mathbf{x}\Big)\mathbf{R},$$

so $(\mathbf{U}|\mathbf{y})\mathbf{A}$ is

$$(\mathbf{U}|\mathbf{y})(\mathbf{T}|\mathbf{x})\mathbf{R} \implies (\mathbf{U}|\mathbf{y})\Big(\tau_{\mathbf{x}}(\mathbf{R}) \,|\, \mathbf{x}\Big)\mathbf{R}.$$

Assume first that $\mathbf{y}$ and $\mathbf{x}$ are distinct, and that $\mathbf{x}$ does not appear in $\mathbf{U}$. Then **CS2** (Metatheorem 6) allows us to write $(\mathbf{U}|\mathbf{y})\mathbf{A}$ as

$$(\mathbf{T}_1|\mathbf{x})(\mathbf{U}|\mathbf{y})\mathbf{R} \implies (\mathbf{T}_2|\mathbf{x})(\mathbf{U}|\mathbf{y})\mathbf{R}$$

where $\mathbf{T}_1$ is $(\mathbf{U}|\mathbf{y})\mathbf{T}$ and $\mathbf{T}_2$ is $(\mathbf{U}|\mathbf{y})\tau_{\mathbf{x}}(\mathbf{R})$. By **CS4** (Metatheorem 8), we can write $\mathbf{T}_2$ as $\tau_{\mathbf{x}}\Big((\mathbf{U}|\mathbf{y})\mathbf{R}\Big)$. If we let $\mathbf{S}$ be $(\mathbf{U}|\mathbf{y})\mathbf{R}$ then $\mathbf{T}_2$ is $\tau_{\mathbf{x}}(\mathbf{S})$, and $(\mathbf{U}|\mathbf{y})\mathbf{A}$ is

$$(\mathbf{T}_1|\mathbf{x})\mathbf{S} \implies (\tau_{\mathbf{x}}(\mathbf{S})|\mathbf{x})\mathbf{S}.$$

Thus $(\mathbf{U}|\mathbf{y})\mathbf{A}$ is produced by S5.

In general, choose a variable $\mathbf{x}'$ distinct from $\mathbf{y}$ and such that $\mathbf{x}'$ does not appear in $\mathbf{R}$ or $\mathbf{U}$. In this case $(\mathbf{T}|\mathbf{x})\mathbf{R}$ is just $(\mathbf{T}|\mathbf{x}')(\mathbf{x}'|\mathbf{x})\mathbf{R}$ by **CS1** (Metatheorem 5), which we can write as $(\mathbf{T}|\mathbf{x}')\mathbf{R}'$ where $\mathbf{R}'$ is $(\mathbf{x}'|\mathbf{x})\mathbf{R}$. Similarly, $\Big(\tau_{\mathbf{x}}(\mathbf{R}) \,|\, \mathbf{x}\Big)\mathbf{R}$ is $\Big(\tau_{\mathbf{x}}(\mathbf{R}) \,|\, \mathbf{x}'\Big)\mathbf{R}'$. By **CS3** (Metatheorem 7) we have $\tau_{\mathbf{x}}(\mathbf{R})$ is the same as $\tau_{\mathbf{x}'}(\mathbf{R}')$, so $\Big(\tau_{\mathbf{x}}(\mathbf{R}) \,|\, \mathbf{x}\Big)\mathbf{R}$ is $\Big(\tau_{\mathbf{x}'}(\mathbf{R}') \,|\, \mathbf{x}'\Big)\mathbf{R}'$. In particular, we can write $\mathbf{A}$ as

$$(\mathbf{T}|\mathbf{x}')\mathbf{R}' \implies \Big(\tau_{\mathbf{x}'}(\mathbf{R}') \,|\, \mathbf{x}'\Big)\mathbf{R}'.$$

Since $\mathbf{x}'$ is not $\mathbf{y}$ and does not appear in $\mathbf{U}$, we get that $(\mathbf{U}|\mathbf{y})\mathbf{A}$ is produced by S5 by the previous argument. □

At this point we have the two central rules for existential quantifiers: the introduction rule (S5) and the elimination rule (Metatheorem 59). This allows us to avoid using $\tau_{\mathbf{x}}$ altogether in our arguments and give more standard proofs of the quantifier laws. So I have replaced some of Bourbaki's proofs in what follows, avoiding $\tau_{\mathbf{x}}$ whenever possible.

We delay Bourbaki's **C29** a bit in order to prove the ∀-elimination rule (**C30**). We have already the ∀-introduction rule (**C27**, Metatheorem 62)), so the elimination rule will give us both major rules for universal quantifiers, and allow us to use standard proofs to establish any other property of universal quantifiers we need.

Metatheorem 65 (∀-elimination, Bourbaki's **C30**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. If* $\mathbf{T}$ *is a term then*

$$(\forall \mathbf{x})\mathbf{R}[\mathbf{x}] \implies \mathbf{R}[\mathbf{T}]$$

is a theorem.

Proof. (Bourbaki's proof uses $\tau_{\mathbf{x}}$). We prove the contrapositive, so suppose $\neg\mathbf{R}[\mathbf{T}]$. Then $(\exists \mathbf{x})(\neg\mathbf{R}[\mathbf{x}])$ by S5. So $\neg\neg(\exists \mathbf{x})(\neg\mathbf{R}[\mathbf{x}])$. Hence $\neg(\forall \mathbf{x})\mathbf{R}[\mathbf{x}]$. □

Metatheorem 66 (Bourbaki's **C29**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Then*

$$\neg(\exists \mathbf{x})\mathbf{R}[\mathbf{x}] \iff (\forall \mathbf{x})\neg\mathbf{R}[\mathbf{x}]$$

is a theorem.

Proof. First assume $\neg(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$ and let $\mathbf{y}$ be a variable that does not appear in $\mathbf{R}[\mathbf{x}]$ and is not a constant. Observe that assuming $\mathbf{R}[\mathbf{y}]$ gives $(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$ by S5, a contradiction. So $\neg\mathbf{R}[\mathbf{y}]$. By ∀-introduction (Metatheorem 62), we have the theorem $(\forall \mathbf{y})\neg\mathbf{R}[\mathbf{y}]$. By **CS8** (Metatheorem 56), we get the theorem $(\forall \mathbf{x})\neg\mathbf{R}[\mathbf{x}]$.

Next assume $(\forall \mathbf{x})\neg\mathbf{R}[\mathbf{x}]$. Then assume $(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$ in an attempt to prove its negation. Let $\mathbf{c}$ be a variable that is not an existing constant and not a variable occurring in $\mathbf{R}[\mathbf{x}]$, and assume $\mathbf{R}[\mathbf{c}]$ (to initiate ∃-elimination). By **C30** (∀-elimination, the previous result) we have $\neg\mathbf{R}[\mathbf{c}]$. This gives a contradiction, which allows us to prove any statement; in particular, we get $\neg(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$. By ∃-elimination and $\implies$-introduction we get $(\exists \mathbf{x})\mathbf{R}[\mathbf{x}] \implies \neg(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$ under our assumption of $(\forall \mathbf{x})\neg\mathbf{R}[\mathbf{x}]$. But this implies $\neg(\exists \mathbf{x})\mathbf{R}[\mathbf{x}]$, as desired. □

Remark. Bourbaki's proof of **C29** uses $\tau_{\mathbf{x}}$ instead.

Metatheorem 67 (First part of Bourbaki's **C31**). *Suppose that* $\mathbf{R}[\mathbf{x}] \implies \mathbf{S}[\mathbf{x}]$ *is a theorem where* $\mathbf{x}$ *is a variable that is not a constant. Then*

$$(\forall \mathbf{x})\mathbf{R}[\mathbf{x}] \implies (\forall \mathbf{x})\mathbf{S}[\mathbf{x}] \qquad \textit{and} \qquad (\exists \mathbf{x})\mathbf{R}[\mathbf{x}] \implies (\exists \mathbf{x})\mathbf{S}[\mathbf{x}]$$

are also theorems.

Proof. Assume $(\forall \mathbf{x})\mathbf{R}[\mathbf{x}]$. Then $\mathbf{R}[\mathbf{x}]$ by ∀-elimination (Metatheorem 65). Thus $\mathbf{S}[\mathbf{x}]$. So $(\forall \mathbf{x})\mathbf{S}[\mathbf{x}]$ by ∀-introduction (Metatheorem 62).

For the second part, start with the contrapositive $\neg\mathbf{S}[\mathbf{x}] \implies \neg\mathbf{R}[\mathbf{x}]$. We get

$$(\forall \mathbf{x})\neg\mathbf{S}[\mathbf{x}] \implies (\forall \mathbf{x})\neg\mathbf{R}[\mathbf{x}]$$

as above. Now apply **C29** (Metatheorem 66) to move negations out in front, and afterwards take the contrapositive. (One can also form a proof from the elimination and introduction rules for ∃). □

Metatheorem 68 (Second part of Bourbaki's **C31**). *Suppose that* $\mathbf{R}[\mathbf{x}] \iff \mathbf{S}[\mathbf{x}]$ *is a theorem where* $\mathbf{x}$ *is a variable that is not a constant. Then*

$$(\forall \mathbf{x})\mathbf{R}[\mathbf{x}] \iff (\forall \mathbf{x})\mathbf{S}[\mathbf{x}] \qquad \textit{and} \qquad (\exists \mathbf{x})\mathbf{R}[\mathbf{x}] \iff (\exists \mathbf{x})\mathbf{S}[\mathbf{x}]$$

are also theorems.

Proof. This is a corollary of the first part of **C31**. □

Metatheorem 69 (Bourbaki's **C32**). *Suppose* $\mathbf{R}[\mathbf{x}]$ *and* $\mathbf{S}[\mathbf{x}]$ *are formulas with designated variable* $\mathbf{x}$*. Then the following are theorems:*

1. $(\forall\mathbf{x})(\mathbf{R}[\mathbf{x}] \wedge \mathbf{S}[\mathbf{x}]) \iff ((\forall\mathbf{x})\mathbf{R}[\mathbf{x}] \wedge (\forall\mathbf{x})\mathbf{S}[\mathbf{x}])$.
2. $(\exists\mathbf{x})(\mathbf{R}[\mathbf{x}] \vee \mathbf{S}[\mathbf{x}]) \iff ((\exists\mathbf{x})\mathbf{R}[\mathbf{x}] \vee (\exists\mathbf{x})\mathbf{S}[\mathbf{x}])$.

Proof. Assume $(\forall\mathbf{x})(\mathbf{R}[\mathbf{x}] \wedge \mathbf{S}[\mathbf{x}])$. Let $\mathbf{y}$ be a variable that is not a constant and that does not appear in $\mathbf{R}[\mathbf{x}]$ or $\mathbf{S}[\mathbf{x}]$. By $\forall$-elimination (Metatheorem 65) we get the theorem $\mathbf{R}[\mathbf{y}] \wedge \mathbf{S}[\mathbf{y}]$. This gives us $\mathbf{R}[\mathbf{y}]$, and thus $(\forall\mathbf{y})\mathbf{R}[\mathbf{y}]$ by $\forall$-introduction (Metatheorem 62). So we have $(\forall\mathbf{x})\mathbf{R}[\mathbf{x}]$ by **CS8** (Metatheorem 56). Similarly we have $(\forall\mathbf{x})\mathbf{S}[\mathbf{x}]$. The converse is similar.

The second equivalence follows from negating both sides of the first equivalence (and negating $\mathbf{R}$ and $\mathbf{S}$). Alternatively, it can be proved by using the introduction and elimination rules for $\exists$. □

Metatheorem 70 (Bourbaki's **C33**). *Suppose* $\mathbf{R}$ *and* $\mathbf{S}[\mathbf{x}]$ *are formulas where the variable* $\mathbf{x}$ *does not appear in* $\mathbf{R}$*. Then the following are theorems:*

1. $(\forall\mathbf{x})(\mathbf{R} \vee \mathbf{S}[\mathbf{x}]) \iff \mathbf{R} \vee (\forall\mathbf{x})\mathbf{S}[\mathbf{x}]$.
2. $(\exists\mathbf{x})(\mathbf{R} \wedge \mathbf{S}[\mathbf{x}]) \iff \mathbf{R} \wedge (\exists\mathbf{x})\mathbf{S}[\mathbf{x}]$.

Proof. These can be proved in a straightforward way with the elimination and introduction rules for $\forall$ and $\exists$. (If $\mathbf{x}$ happens to be a constant, then we need to exercise some care in the first proof and choose a new variable as in the previous proof. Bourbaki has different proofs.) □

Now we consider results related to multiple quantifiers.

Metatheorem 71. *Suppose* $\mathbf{R}$*, also written* $\mathbf{R}[\mathbf{x},\mathbf{y}]$*, is a formula with two distinct designated variables* $\mathbf{x}$ *and* $\mathbf{y}$*. Suppose that* $\mathbf{x}'$ *and* $\mathbf{y}'$ *are distinct variables that are each distinct from* $\mathbf{x}$ *and* $\mathbf{y}$ *and any other variable appearing in* $\mathbf{R}$*. Then*

$$(\forall\mathbf{y})(\forall\mathbf{x})\mathbf{R}[\mathbf{x},\mathbf{y}] \quad \textit{is the same as} \quad (\forall\mathbf{y}')(\forall\mathbf{x}')\mathbf{R}[\mathbf{x}',\mathbf{y}'],$$

which is the same as $(\forall\mathbf{y}')(\forall\mathbf{x}')(\mathbf{y}'|\mathbf{y})(\mathbf{x}'|\mathbf{x})\mathbf{R}$*. Similarly,*

$$(\exists\mathbf{y})(\exists\mathbf{x})\mathbf{R}[\mathbf{x},\mathbf{y}] \quad \textit{is the same as} \quad (\exists\mathbf{y}')(\exists\mathbf{x}')\mathbf{R}[\mathbf{x}',\mathbf{y}'],$$

$$(\forall\mathbf{y})(\exists\mathbf{x})\mathbf{R}[\mathbf{x},\mathbf{y}] \quad \textit{is the same as} \quad (\forall\mathbf{y}')(\exists\mathbf{x}')\mathbf{R}[\mathbf{x}',\mathbf{y}'],$$

and

$$(\exists\mathbf{y})(\forall\mathbf{x})\mathbf{R}[\mathbf{x},\mathbf{y}] \quad \textit{is the same as} \quad (\exists\mathbf{y}')(\forall\mathbf{x}')\mathbf{R}[\mathbf{x}',\mathbf{y}'].$$

Proof. Since $\mathbf{x}'$ does not appear in $\mathbf{R}$, Metatheorem 56 (**CS8**) tells us that $(\forall\mathbf{x})\mathbf{R}$ is the same as $(\forall\mathbf{x}')(\mathbf{x}'|\mathbf{x})\,\mathbf{R}$. By definition of $\forall$, the variables appearing in $(\forall\mathbf{x})\mathbf{R}$ are the same as the variables appearing in $\mathbf{R}$, except that $\mathbf{x}$ does not appear in $(\forall\mathbf{x})\mathbf{R}$. In particular, $\mathbf{y}'$ does not appear in $(\forall\mathbf{x})\mathbf{R}$. So, by **CS8** again, the formula $(\forall\mathbf{y})(\forall\mathbf{x})\mathbf{R}$ is the same as $(\forall\mathbf{y}')(\mathbf{y}'|\mathbf{y})(\forall\mathbf{x})\mathbf{R}$, which in turn is the same as $(\forall\mathbf{y}')(\mathbf{y}'|\mathbf{y})(\forall\mathbf{x}')(\mathbf{x}'|\mathbf{x})\mathbf{R}$. Since $\mathbf{x}'$ is distinct from $\mathbf{y}$ and $\mathbf{y}'$, we can invoke rule **CS9** (Metatheorem 57) to infer that this formula is the same as $(\forall \mathbf{y}')(\forall \mathbf{x}')(\mathbf{y}'|\mathbf{y})(\mathbf{x}'|\mathbf{x})\mathbf{R}$, which is just $(\forall \mathbf{x}')(\forall \mathbf{y}')\mathbf{R}[\mathbf{x}', \mathbf{y}']$.

The other statements are handled similarly. $\square$

Metatheorem 72 (Bourbaki's **C34**)**.** *Suppose* **R** *is a formula and* **x** *and* **y** *are variables. Then the following are theorems:*

1. $(\forall \mathbf{x})(\forall \mathbf{y})\mathbf{R} \iff (\forall \mathbf{y})(\forall \mathbf{x})\mathbf{R}$.
2. $(\exists \mathbf{x})(\exists \mathbf{y})\mathbf{R} \iff (\exists \mathbf{y})(\exists \mathbf{x})\mathbf{R}$.
3. $(\exists \mathbf{x})(\forall \mathbf{y})\mathbf{R} \implies (\forall \mathbf{y})(\exists \mathbf{x})\mathbf{R}$.

Proof. The first statement can be easily proved with the elimination and introduction rules for $\forall$, assuming **x** and **y** are not constants. The case where **x** and **y** are the same variable can handled separately, and is immediate even if the variables are constants. If **x** or **y** happens to be a constant, and are distinct, then we can use Metatheorem 71 and introduce new variables to reduce to the case where the variables are not constants. (Bourbaki does not use Metatheorem 71 but instead first proves this in the weaker theory where the simple axioms are removed, so all variables are non-constants. See **C4**, Metatheorem 20.)

The second statement can be proved by applying the first statement to $\neg\mathbf{R}$ and applying **C29** and **C31**. A more direct proof using the elimination and introduction rules for $\exists$ can be given as well.

The third statement can be first proved in the case where **x** and **y** are distinct and nonconstant using $\exists$-elimination (as in Metatheorem 58), $\forall$-elimination, and the introduction rules for $\exists$ and $\forall$. If the variables are distinct and either happens to be a constant, we reduce to the nonconstant case using the same strategy as in the proof of the first statement. (We can handle the case where **x** and **y** are the same variable separately). $\square$

6 Bounded quantifiers (Bourbaki's §4.4)

We define quantifiers relative to a property $A(x)$ that express the idea that "there exists an x with property $A(x)$ such that ...", and "every x with property $A(x)$ is such that ...". These are the most common quantifiers one actually sees in mathematics. Bourbaki calls these "typical quantifiers" but I will use the more common term: "bounded quantifiers".

Definition 15 (Bounded quantifiers)**.** Suppose **A** and **R** are formulas and **x** is a variable. Then

$$(\exists_{\mathbf{A}} \mathbf{x})\mathbf{R} \quad \text{is short for} \quad (\exists\, \mathbf{x})(\mathbf{A} \wedge \mathbf{R}),$$

and

$$(\forall_{\mathbf{A}} \mathbf{x})\mathbf{R} \quad \text{is short for} \quad \neg(\exists_{\mathbf{A}} \mathbf{x})(\neg\mathbf{R}).$$

The following follows from the earlier formative criteria (CF):

Metatheorem 73 (Bourbaki's **CF12**)**.** *Suppose* **A** *and* **R** *are formulas and* **x** *is a variable. Then* $(\exists_{\mathbf{A}} \mathbf{x})\mathbf{R}$ *and* $(\forall_{\mathbf{A}} \mathbf{x})\mathbf{R}$ *are formulas.*

Metatheorem 74. *Suppose* $\mathbf{A}$ *and* $\mathbf{R}$ *are formulas and* $\mathbf{x}$ *is a variable. Then* $\mathbf{x}$ *does not appear in the formulas* $(\exists_{\mathbf{A}}\mathbf{x})\mathbf{R}$ *and* $(\forall_{\mathbf{A}}\mathbf{x})\mathbf{R}$*. Aside from* $\mathbf{x}$*, the same variables appear in* $\mathbf{A} \wedge \mathbf{R}$*,* $(\exists_{\mathbf{A}}\mathbf{x})\mathbf{R}$ *and* $(\forall_{\mathbf{A}}\mathbf{x})\mathbf{R}$*.*

From the earlier criteria of substitution (CS) we have the following two new criteria of substitution:

Metatheorem 75 (Bourbaki's **CS10**)**.** *Suppose that the variable* $\mathbf{y}$ *is not a parameter in the formulas* $\mathbf{A}[\mathbf{x}]$ *and* $\mathbf{R}[\mathbf{x}]$*. Then* $(\exists_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is just* $(\exists_{\mathbf{A}[\mathbf{y}]}\mathbf{y})\mathbf{R}[\mathbf{y}]$*. Similarly,* $(\forall_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is just* $(\forall_{\mathbf{A}[\mathbf{y}]}\mathbf{y})\mathbf{R}[\mathbf{y}]$*.*

Proof. This follows from **CS8** (Metatheorem 56). ☐

Metatheorem 76 (Bourbaki's **CS11**)**.** *Suppose that* $\mathbf{x}$ *and* $\mathbf{y}$ *are distinct variables. Suppose that* $\mathbf{A}[\mathbf{y}]$ *and* $\mathbf{R}[\mathbf{y}]$ *are formulas with designated variable* $\mathbf{y}$ *(and which might include* $\mathbf{x}$*), and suppose* $\mathbf{U}$ *is a term where* $\mathbf{x}$ *does not appear in* $\mathbf{U}$*. Then*

$$(\mathbf{U}|\mathbf{y})\Big((\exists_{\mathbf{A}[\mathbf{y}]}\mathbf{x})\,\mathbf{R}[\mathbf{y}]\Big) \text{ is the same as } (\exists_{\mathbf{A}[\mathbf{U}]}\mathbf{x})\,\mathbf{R}[\mathbf{U}].$$

and

$$(\mathbf{U}|\mathbf{y})\Big((\forall_{\mathbf{A}[\mathbf{y}]}\mathbf{x})\,\mathbf{R}[\mathbf{y}]\Big) \text{ is the same as } (\forall_{\mathbf{A}[\mathbf{U}]}\mathbf{x})\,\mathbf{R}[\mathbf{U}].$$

Proof. This follows from **CS9** (Metatheorem 57). ☐

Metatheorem 77 ($\exists_{\mathbf{A}}$-introduction)**.** *Suppose that* $\mathbf{A}[\mathbf{x}]$ *and* $\mathbf{R}[\mathbf{x}]$ *are formulas with designated variable* $\mathbf{x}$*, and that* $\mathbf{T}$ *is a term. If* $\mathbf{A}[\mathbf{T}]$ *and* $\mathbf{R}[\mathbf{T}]$ *are theorems then so is* $(\exists_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\,\mathbf{R}[\mathbf{x}]$*.*

Proof. This follows from S5 and the definition of $\exists_{\mathbf{A}[\mathbf{x}]}$. ☐

Metatheorem 78 ($\exists_{\mathbf{A}}$-elimination rule)**.** *Let* $\mathbf{x}$ *be a variable and let* $\mathbf{A}[\mathbf{x}], \mathbf{R}[\mathbf{x}]$ *and* $\mathbf{B}$ *be formulas. Assume that* (1) $(\exists_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{R}[\mathbf{x}]$ *is a theorem of* $\mathcal{T}$*,* (2) $\mathbf{c}$ *is an unused variable, or at least it is not a constant and does not appear in* $\mathbf{B}, \mathbf{A}[\mathbf{x}]$ *and* $\mathbf{R}[\mathbf{x}]$*,* (3) $\mathbf{A}[\mathbf{c}] \wedge \mathbf{R}[\mathbf{c}] \implies \mathbf{B}$ *is a theorem. Then* $\mathbf{B}$ *is a theorem.*

Proof. This follows from the $\exists$-elimination rule (Metatheorem 59). ☐

Metatheorem 79 (Bourbaki's **C35**)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{R}$ *are formulas and* $\mathbf{x}$ *is a variable. Then*

$$(\forall_{\mathbf{A}}\mathbf{x})\mathbf{R} \iff (\forall \mathbf{x})(\mathbf{A} \implies \mathbf{R})$$

is a theorem.

Proof. Using **CS10** (Metatheorem 75) and **CS8** (Metatheorem 56) we can reduce to the case where $\mathbf{x}$ is not a constant (or just work in a weaker theory without constants). Start with

$$\mathbf{A} \wedge (\neg \mathbf{R}) \iff \neg(\mathbf{A} \implies \mathbf{R}).$$

So, by **C31** (Metatheorem 68),

$$\exists \mathbf{x}(\mathbf{A} \wedge (\neg \mathbf{R})) \iff \exists \mathbf{x}\, \neg(\mathbf{A} \implies \mathbf{R}).$$

Thus
$$\neg\exists \mathbf{x}(\mathbf{A} \wedge (\neg \mathbf{R})) \iff \neg\exists \mathbf{x}\,\neg(\mathbf{A} \implies \mathbf{R})$$
as desired. □

Remark. It might be preferable to use the above **C35** as the actual definition of $\forall_{\mathbf{A}}$ since it is more closely tied to the introduction and elimination rules.

Metatheorem 80 ($\forall_{\mathbf{A}}$-introduction, or Bourbaki's **C36**)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{R}$ *are formulas in a theory* $\mathcal{T}$ *and that* $\mathbf{x}$ *is a variable that is not a constant of* $\mathcal{T}$*. Let* $\mathcal{T}'$ *be the theory obtained by adding the formula* $\mathbf{A}$ *to the simple axioms of* $\mathcal{T}$*. If* $\mathbf{R}$ *is a theorem of* $\mathcal{T}'$ *then* $(\forall_{\mathbf{A}} \mathbf{x})\, \mathbf{R}$ *is a theorem of* $\mathcal{T}$*.*

Proof. By the $\implies$-introduction rule $\mathbf{A} \implies \mathbf{R}$ is a theorem of $\mathcal{T}$ (See Metatheorem 36, **C14**). By the $\forall$-introduction rule $\forall \mathbf{x}\,(\mathbf{A} \implies \mathbf{R})$ is a theorem of $\mathcal{T}$ (See Metatheorem 62, **C27**). Now use **C35** above. □

Metatheorem 81 ($\forall_{\mathbf{A}}$-elimination rule)**.** *Suppose* $\mathbf{A}[\mathbf{x}]$ *and* $\mathbf{R}[\mathbf{x}]$ *are formulas with designated variable* $\mathbf{x}$*, and that* $\mathbf{T}$ *is a term. If* $(\forall_{\mathbf{A}[\mathbf{x}]} \mathbf{x})\, \mathbf{R}[\mathbf{x}]$ *and* $\mathbf{A}[\mathbf{T}]$ *are theorems then so is* $\mathbf{R}[\mathbf{T}]$*.*

Proof. We have $\forall \mathbf{x}\,(\mathbf{A}[\mathbf{x}] \implies \mathbf{R}[\mathbf{x}])$ by **C35** (Metatheorem 79). Thus, by $\forall$-elimination, $\mathbf{A}[\mathbf{T}] \implies \mathbf{R}[\mathbf{T}]$ is a theorem. Now use modus ponens. □

Here is a form of reductio ad absurdum. I am not sure that it is actually needed, so I will give it as an exercise:

Exercise 2 (Bourbaki's **C37**)**.** Suppose $\mathbf{A}$ and $\mathbf{R}$ are formulas in a theory $\mathcal{T}$ and that $\mathbf{x}$ is a variable that is not a constant of $\mathcal{T}$. Let $\mathcal{T}'$ be the theory obtained by adding the formulas $\mathbf{A}$ and $\neg\mathbf{R}$ to the simple axioms of $\mathcal{T}$. Show that if $\mathcal{T}'$ is contradictory then $(\forall_{\mathbf{A}} \mathbf{x})\, \mathbf{R}$ is a theorem of $\mathcal{T}$.

Next follows six results concerning bounded quantifiers. Bourbaki leaves the proofs to the reader. They can all be proved with the introduction and elimination rules for bounded quantifiers, or by converting to non-bounded quantifiers.

Metatheorem 82 (Bourbaki's **C38**)**.** *Let* $\mathbf{A}$ *and* $\mathbf{R}$ *be formulas and let* $\mathbf{x}$ *be a variable. Then the following are theorems:*
$$\neg(\forall_{\mathbf{A}} \mathbf{x})\mathbf{R} \iff (\exists_{\mathbf{A}} \mathbf{x})(\neg\mathbf{R}) \quad \textit{and} \quad \neg(\exists_{\mathbf{A}} \mathbf{x})\mathbf{R} \iff (\forall_{\mathbf{A}} \mathbf{x})(\neg\mathbf{R}).$$

Metatheorem 83 (Bourbaki's **C39** part 1)**.** *Let* $\mathbf{A}, \mathbf{R}$ *and* $\mathbf{S}$ *be formulas and let* $\mathbf{x}$ *be a variable that is not a constant. If* $\mathbf{A} \implies (\mathbf{R} \implies \mathbf{S})$ *is a theorem then the following are also theorems:*
$$(\exists_{\mathbf{A}} \mathbf{x})\mathbf{R} \implies (\exists_{\mathbf{A}} \mathbf{x})\mathbf{S} \quad \textit{and} \quad (\forall_{\mathbf{A}} \mathbf{x})\mathbf{R} \implies (\forall_{\mathbf{A}} \mathbf{x})\mathbf{S}.$$

Metatheorem 84 (Bourbaki's **C39** part 2)**.** *Let* $\mathbf{A}, \mathbf{R}$ *and* $\mathbf{S}$ *be formulas and let* $\mathbf{x}$ *be a variable that is not a constant. If* $\mathbf{A} \implies (\mathbf{R} \iff \mathbf{S})$ *is a theorem then the following are also theorems:*
$$(\exists_{\mathbf{A}} \mathbf{x})\mathbf{R} \iff (\exists_{\mathbf{A}} \mathbf{x})\mathbf{S} \quad \textit{and} \quad (\forall_{\mathbf{A}} \mathbf{x})\mathbf{R} \iff (\forall_{\mathbf{A}} \mathbf{x})\mathbf{S}.$$

Metatheorem 85 (Bourbaki's **C40**). *Let* $\mathbf{A}, \mathbf{R}$ *and* $\mathbf{S}$ *be formulas and let* $\mathbf{x}$ *be a variable. If* $\mathbf{A} \implies (\mathbf{R} \iff \mathbf{S})$ *is a theorem then the following are also theorems:*

$$(\forall_{\mathbf{A}}\mathbf{x})(\mathbf{R} \wedge \mathbf{S}) \iff (\forall_{\mathbf{A}}\mathbf{x})(\mathbf{R}) \wedge (\forall_{\mathbf{A}}\mathbf{x})(\mathbf{S})$$

$$(\exists_{\mathbf{A}}\mathbf{x})(\mathbf{R} \vee \mathbf{S}) \iff (\exists_{\mathbf{A}}\mathbf{x})(\mathbf{R}) \vee (\exists_{\mathbf{A}}\mathbf{x})(\mathbf{S}).$$

Metatheorem 86 (Bourbaki's **C41**). *Let* $\mathbf{A}[\mathbf{x}]$ *and* $\mathbf{S}[\mathbf{x}]$ *be formulas with designated variable* $\mathbf{x}$ *and let* $\mathbf{R}$ *be a formula not containing* $\mathbf{x}$*. Then the following are also theorems:*

$$(\forall_{\mathbf{A}[\mathbf{x}]}\mathbf{x})(\mathbf{R} \vee \mathbf{S}[\mathbf{x}]) \iff \mathbf{R} \vee (\forall_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{S}[\mathbf{x}]$$

$$(\exists_{\mathbf{A}[\mathbf{x}]}\mathbf{x})(\mathbf{R} \wedge \mathbf{S}[\mathbf{x}]) \iff \mathbf{R} \wedge (\exists_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{S}[\mathbf{x}]$$

$$(\forall_{\mathbf{A}[\mathbf{x}]}\mathbf{x})(\mathbf{R} \implies \mathbf{S}[\mathbf{x}]) \iff (\mathbf{R} \implies (\forall_{\mathbf{A}[\mathbf{x}]}\mathbf{x})\mathbf{S}[\mathbf{x}]).$$

Metatheorem 87 (Bourbaki's **C42**). *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{R}$ *be formulas. Let* $\mathbf{x}$ *be a variable not appearing in* $\mathbf{B}$ *and let* $\mathbf{y}$ *be a variable not appearing in* $\mathbf{A}$*. Then the following are also theorems:*

$$(\forall_{\mathbf{A}}\mathbf{x})(\forall_{\mathbf{B}}\mathbf{y})\mathbf{R} \iff (\forall_{\mathbf{B}}\mathbf{y})(\forall_{\mathbf{A}}\mathbf{x})\mathbf{R}$$

$$(\exists_{\mathbf{A}}\mathbf{x})(\exists_{\mathbf{B}}\mathbf{y})\mathbf{R} \iff (\exists_{\mathbf{B}}\mathbf{y})(\exists_{\mathbf{A}}\mathbf{x})\mathbf{R}$$

$$(\exists_{\mathbf{A}}\mathbf{x})(\forall_{\mathbf{B}}\mathbf{y})\mathbf{R} \implies (\forall_{\mathbf{B}}\mathbf{y})(\exists_{\mathbf{A}}\mathbf{x})\mathbf{R}.$$

7 Logic with equality (§5.1 and §5.2 in Bourbaki)

Above we assumed only that we work in a language that includes axiom schemes S1 to S5. We did not assume the existence of relational or function symbols, or any simple axioms. In this section we add the assumption that the language has a binary relational symbol $=$, for which we adopt as a convenient abbreviation the infix notation. In particular we write $\mathbf{T} = \mathbf{U}$ instead of the functional notation $= \mathbf{T}\mathbf{U}$. We often write $\mathbf{T} \neq \mathbf{U}$ for $\neg(\mathbf{T} = \mathbf{U})$.

Related to $=$ we adopt two additional schemes S6 and S7 (and we will continue to assume S1 to S5):

S6. Suppose $\mathbf{T}$ and $\mathbf{U}$ are terms and $\mathbf{R}[\mathbf{x}]$ is a formula with designated variable $\mathbf{x}$. Then

$$(\mathbf{T} = \mathbf{U}) \implies \big(\mathbf{R}[\mathbf{T}] \iff \mathbf{R}[\mathbf{U}]\big)$$

is an axiom.

S7. Suppose $\mathbf{R}$ and $\mathbf{S}$ are formulas and $\mathbf{x}$ is a variable. Then

$$\big((\forall \mathbf{x})(\mathbf{R} \iff \mathbf{S})\big) \implies \big(\tau_{\mathbf{x}}(\mathbf{R}) = \tau_{\mathbf{x}}(\mathbf{S})\big)$$

is an axiom.[11]

[11]In particular, τ denotes an "extensional choice", not merely some "intensional choice" that we make separately for each formula.

Recall that we require that if $\mathbf{A}$ is produced by an axiom scheme then so should $(\mathbf{T}|\mathbf{y})\mathbf{A}$ for any term $\mathbf{T}$ and variable $\mathbf{y}$. So before we can use S6 and S7 we should verify that they verify this requirement:

Metatheorem 88. *Axiom scheme* S6 *satisfies the scheme requirement: if* $\mathbf{A}$ *is produced by* S6 *then so is* $(\mathbf{S}|\mathbf{y})\mathbf{A}$ *for any term* $\mathbf{S}$ *and variable* $\mathbf{y}$*.*

Proof. Any $\mathbf{A}$ produced by S6 is of the form

$$(\mathbf{T} = \mathbf{U}) \implies \big((\mathbf{T}|\mathbf{x})\mathbf{R} \iff (\mathbf{U}|\mathbf{x})\mathbf{R}\big)$$

where $\mathbf{R}, \mathbf{T}, \mathbf{U}$ are terms and $\mathbf{x}$ is a variable. By **CS1** (Metatheorem 5), if $\mathbf{x}'$ is a variable not in $\mathbf{R}$ then $(\mathbf{T}|\mathbf{x})\mathbf{R}$ is the same as $(\mathbf{T}|\mathbf{x}')\mathbf{R}'$ where $\mathbf{R}'$ is $(\mathbf{x}'|\mathbf{x})\mathbf{R}$. Similarly, $(\mathbf{U}|\mathbf{x})\mathbf{R}$ is the same as $(\mathbf{U}|\mathbf{x}')\mathbf{R}'$. So we can replace $\mathbf{R}$ and $\mathbf{x}$ if necessary to assume that $\mathbf{x}$ is not equal to $\mathbf{y}$, and that $\mathbf{x}$ is not in $\mathbf{S}$.

Observe that $(\mathbf{S}|\mathbf{y})\mathbf{A}$ is

$$((\mathbf{S}|\mathbf{y})\mathbf{T} = (\mathbf{S}|\mathbf{y})\mathbf{U}) \implies \big((\mathbf{S}|\mathbf{y})(\mathbf{T}|\mathbf{x})\mathbf{R} \iff (\mathbf{S}|\mathbf{y})(\mathbf{U}|\mathbf{y})\mathbf{R}\big),$$

which by **CS2** (Metatheorem 6) is

$$((\mathbf{S}|\mathbf{y})\mathbf{T} = (\mathbf{S}|\mathbf{y})\mathbf{U}) \implies \big((\mathbf{T}'|\mathbf{x})(\mathbf{S}|\mathbf{y})\mathbf{R} \iff (\mathbf{U}'|\mathbf{x})(\mathbf{S}|\mathbf{y})\mathbf{R}\big)$$

where $\mathbf{T}'$ is $(\mathbf{S}|\mathbf{y})\mathbf{T}$ and $\mathbf{U}'$ is $(\mathbf{S}|\mathbf{y})\mathbf{U}$. So if $\mathbf{R}'$ is $(\mathbf{S}|\mathbf{y})\mathbf{R}$ then $(\mathbf{S}|\mathbf{y})\mathbf{A}$ can be written as

$$(\mathbf{T}' = \mathbf{U}') \implies \big((\mathbf{T}'|\mathbf{x})\mathbf{R}' \iff (\mathbf{U}'|\mathbf{x})\mathbf{R}'\big)$$

which is produced by S6. □

Metatheorem 89. *Axiom scheme* S7 *satisfies the scheme requirement: if* $\mathbf{A}$ *is produced by* S7 *then so is* $(\mathbf{T}|\mathbf{y})\mathbf{A}$ *for any term* $\mathbf{T}$ *and variable* $\mathbf{y}$*.*

Proof. Any $\mathbf{A}$ produced by S7 is of the form

$$\big((\forall \mathbf{x})(\mathbf{R} \iff \mathbf{S})\big) \implies \big(\tau_{\mathbf{x}}(\mathbf{R}) = \tau_{\mathbf{x}}(\mathbf{S})\big)$$

By **CS3** (Metatheorem 7) and By **CS8** (Metatheorem 56), if $\mathbf{x}'$ is a variable not appearing in $\mathbf{R}$ or $\mathbf{S}$ then $\mathbf{A}$ can be written as

$$\big((\forall \mathbf{x}')((\mathbf{x}'|\mathbf{x})\mathbf{R} \iff (\mathbf{x}'|\mathbf{x})\mathbf{S})\big) \implies \big(\tau_{\mathbf{x}'}((\mathbf{x}'|\mathbf{x})\mathbf{R}) = \tau_{\mathbf{x}'}((\mathbf{x}'|\mathbf{x})\mathbf{S})\big).$$

So by replacing $\mathbf{x}$ with a suitable $\mathbf{x}'$ if necessary, and then replacing $\mathbf{R}$ with $(\mathbf{x}'|\mathbf{x})\mathbf{R}$ and $\mathbf{S}$ with $(\mathbf{x}'|\mathbf{x})\mathbf{S}$, we can assume that $\mathbf{x}$ is distinct from $\mathbf{y}$ and that $\mathbf{x}$ is not in $\mathbf{T}$.

Note that, by **CS9** (Metatheorem 57),

$$(\mathbf{T}|\mathbf{y})\big((\forall \mathbf{x})(\mathbf{R} \iff \mathbf{S})\big) \quad \text{is just} \quad (\forall \mathbf{x})((\mathbf{T}|\mathbf{y})\mathbf{R} \iff (\mathbf{T}|\mathbf{y})\mathbf{S}).$$

Similarly, observe that $(\mathbf{T}|\mathbf{y})\tau_{\mathbf{x}}(\mathbf{R})$ is $\tau_{\mathbf{x}}((\mathbf{T}|\mathbf{y})\mathbf{R})$ by **CS4** (Metatheorem 8), and that $(\mathbf{T}|\mathbf{y})\tau_{\mathbf{x}}(\mathbf{S})$ is $\tau_{\mathbf{x}}((\mathbf{T}|\mathbf{y})\mathbf{S})$.

So if $\mathbf{R}'$ is $(\mathbf{T}|\mathbf{y})\mathbf{R}$ and if $\mathbf{S}'$ is $(\mathbf{T}|\mathbf{y})\mathbf{S}$ then $(\mathbf{T}|\mathbf{y})\mathbf{A}$ is

$$\big((\forall \mathbf{x})(\mathbf{R}' \iff \mathbf{S}')\big) \implies \big(\tau_{\mathbf{x}}(\mathbf{R}') = \tau_{\mathbf{x}}(\mathbf{S}')\big)$$

which is produced by S7. □

Now having gotten the formal preliminaries out of the way, we can derive the standard results about equality from S6 and S7. The following few results assert specific theorems (asserts specific formulas are theorems) so we will label them as "theorems", not as "metatheorems".

Theorem 1 (Reflexivity, Bourbaki's Theorem 1)**.** *Equality is reflexive:*

$$(\forall x)(x = x).$$

Proof. Suppose $\neg(\forall x)(x = x)$, or equivalently $(\exists x)(x \neq x)$. By definition of the existential quantifier (Definition 13),

$$\tau_x(x \neq x) \neq \tau_x(x \neq x).$$

However by S7,

$$\tau_x(x \neq x) = \tau_x(x \neq x),$$

a contradiction. □

Remark. In Bourbaki's original version of reflexivity the theorem is stated simply as $x = x$, without the quantifier. However, Bourbaki assumes that we are working in a theory with no constants, and so in particular x is not a constant. So $x = x$ is is equivalent to $(\forall x)(x = x)$. I think the statement $(\forall x)(x = x)$ is better since we do not have to assume we are in a theory without constants.

Interestingly, Bourbaki proves $(\forall x)(x = x)$ along the way to proving $x = x$. (Even so, their proof is a bit different than mine. As in several places in this document, I have felt free to rework the proofs a bit.)

Bourbaki does not use quantifiers for the symmetry and transitivity laws either, but I prefer to include them.

Theorem 2 (Symmetry, Bourbaki's Theorem 2)**.** *Equality is symmetric:*

$$\forall x\, \forall y\, (x = y \implies y = x).$$

Proof. Changing variables if necessary, we can assume that x and y are not constants (Metatheorem 71 allows us to change variables if x or y is a constant). We assume $x = y$. Let $\mathbf{R}[z]$ be the formula $z = x$. By reflexivity, we have $\mathbf{R}[x]$, and by S6 we have $\mathbf{R}[x] \iff \mathbf{R}[y]$. So $\mathbf{R}[y]$. In other words, $y = x$. Now apply $\implies$-introduction, and $\forall$-introduction twice. □

Theorem 3 (Transitivity, Bourbaki's Theorem 3)**.** *Equality is transitive:*

$$\forall x\, \forall y\, \forall z\, ((x = y) \wedge (y = z) \implies x = z).$$

Proof. Changing variables if necessary, we can assume that x, y, z are not constants (using a generalization of Metatheorem 71 to do so). Assume $x = y$ and $y = z$. Let $\mathbf{R}[w]$ be the formula $x = w$. By assumption we have $\mathbf{R}[y]$. Thus by $y = z$ and S6 we have $\mathbf{R}[z]$. In other words, $x = z$. Now apply $\implies$-introduction, and $\forall$-introduction three times. □

The next result is an important substitution property of equality:

Metatheorem 90 (Bourbaki's **C44**). *Suppose* $\mathbf{V}[\mathbf{x}]$ *is a term with designated variable* $\mathbf{x}$ *and suppose* $\mathbf{T}$ *and* $\mathbf{U}$ *are terms. Then the following is a theorem:*

$$(\mathbf{T} = \mathbf{U}) \implies (\mathbf{V}[\mathbf{T}] = \mathbf{V}[\mathbf{U}]).$$

Proof. Assume $\mathbf{T} = \mathbf{U}$. Let $\mathbf{R}[\mathbf{y}]$ be the formula $\mathbf{V}[\mathbf{T}] = \mathbf{V}[\mathbf{y}]$ where $\mathbf{y}$ is a variable not occurring in $\mathbf{V}[\mathbf{x}]$ or $\mathbf{T}$. Note that $\mathbf{R}[\mathbf{T}]$ holds by the reflexive law of equality. Thus $\mathbf{R}[\mathbf{U}]$ holds by S6. Thus $\mathbf{V}[\mathbf{T}] = \mathbf{V}[\mathbf{U}]$. □

The next result is given as an exercise (since I am not sure it is needed anywhere in this document):

Exercise 3 (Bourbaki's **C43**). Suppose $\mathbf{R}[\mathbf{x}]$ is a formula with designated variable $\mathbf{x}$ and suppose $\mathbf{T}$ and $\mathbf{U}$ are terms. Show that the following is a theorem:

$$\big((\mathbf{T} = \mathbf{U}) \wedge \mathbf{R}[\mathbf{T}]\big) \iff \big((\mathbf{T} = \mathbf{U}) \wedge \mathbf{R}[\mathbf{U}]\big).$$

8 Unique existence (Bourbaki §5.3)

Definition 16. Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$. The formula "there exists at most one $\mathbf{x}$ such that $\mathbf{R}[\mathbf{x}]$" is the formula

$$(\forall \mathbf{y})(\forall \mathbf{z})\Big(\mathbf{R}[\mathbf{y}] \wedge \mathbf{R}[\mathbf{z}] \implies \mathbf{y} = \mathbf{z}\Big)$$

where $\mathbf{y}$ and $\mathbf{z}$ are distinct variables that are not parameters of $\mathbf{R}[\mathbf{x}]$.

Remark. Bourbaki also requires that $\mathbf{y}$ and $\mathbf{z}$ be distinct from $\mathbf{x}$, but that is clearly not necessary.

Metatheorem 91. *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. The formula "there exists at most one* $\mathbf{x}$ *such that* $\mathbf{R}[\mathbf{x}]$*" is independent of the choice of variables* $\mathbf{y}$ *and* $\mathbf{z}$*.*

Proof. Given two suitable choices of variables $\mathbf{y}, \mathbf{z}$ and $\mathbf{y}', \mathbf{z}'$, consider a third suitable choice $\mathbf{y}'', \mathbf{z}''$ where $\mathbf{y}''$ and $\mathbf{z}''$ are each distinct from $\mathbf{y}, \mathbf{z}, \mathbf{y}', \mathbf{z}'$. Metatheorem 71 implies that both initial choices give the same formula as the third. □

Metatheorem 92. *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. The formula "there exists at most one* $\mathbf{x}$ *such that* $\mathbf{R}[\mathbf{x}]$*" does not contain the variable* $\mathbf{x}$*, but otherwise contains the same variables as* $\mathbf{R}[\mathbf{x}]$*. In other words, its variables are the parameters of* $\mathbf{R}[\mathbf{x}]$*.*

Proof. This follows from the definition and Metatheorem 55 (**CF11**). □

Remark. Bourbaki says $\mathbf{R}[\mathbf{x}]$ is *single-valued* if "there exists at most one $\mathbf{x}$ such that $\mathbf{R}[\mathbf{x}]$" is a theorem. In other words, we can regard "$\mathbf{R}[\mathbf{x}]$ is single-valued (in $\mathbf{x}$)" as synonymous with "there exists at most one $\mathbf{x}$ such that $\mathbf{R}[\mathbf{x}]$".

The proof of the following is straightforward:

Metatheorem 93 (part of Bourbaki **C45**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. If* $\mathbf{T}$ *is a term that does not contain* $\mathbf{x}$ *and if*

$$(\forall\mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \implies \mathbf{x} = \mathbf{T}\Big)$$

is a theorem, then "there exists at most one $\mathbf{x}$ *such that* $\mathbf{R}[\mathbf{x}]$*" is a theorem.*

Metatheorem 94 (part of Bourbaki **C45**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Then the following is a theorem:*

$$\text{"there exists at most one } \mathbf{x} \text{ such that } \mathbf{R}[\mathbf{x}]\text{"} \iff (\forall\mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \implies (\mathbf{x} = \tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}]))\Big).$$

Proof. One direction comes from the previous result. So we make the hypotheses "there exists at most one $\mathbf{x}$ such that $\mathbf{R}[\mathbf{x}]$" and $\mathbf{R}[\mathbf{w}]$ where $\mathbf{w}$ is not a constant nor a parameter of $\mathbf{R}[\mathbf{x}]$. From there the proof is straightforward (using the original form of S5, and change of bound variables from $\mathbf{w}$ to $\mathbf{x}$ using **CS8**, Metatheorem 56). □

Definition 17. Let $\mathbf{R}$ be a formula. The formula "there exists exactly one $\mathbf{x}$ such that $\mathbf{R}$" is the formula

$$((\exists\mathbf{x})\mathbf{R}) \wedge (\text{"there exists at most one } \mathbf{x} \text{ such that } \mathbf{R}\text{"}).$$

We usually write this formula as $(\exists!\,\mathbf{x})\mathbf{R}$.

Remark. If $(\exists!\,\mathbf{x})\mathbf{R}[\mathbf{x}]$ is a theorem, then Bourbaki calls $\mathbf{R}[\mathbf{x}]$ a "functional relation in $\mathbf{x}$". In other words, we can regard "$\mathbf{R}[\mathbf{x}]$ is a function relation $\mathbf{x}$" as synonymous with "$(\exists!\,\mathbf{x})\mathbf{R}[\mathbf{x}]$". (Recall that Bourbaki uses the term "relation" for our term "formula").

Metatheorem 95 (part of Bourbaki **C46**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. If* $\mathbf{T}$ *is a term that does not contain* $\mathbf{x}$ *and if*

$$(\forall\mathbf{x})\big(\mathbf{R}[\mathbf{x}] \iff (\mathbf{x} = \mathbf{T})\big)$$

is a theorem, then $\mathbf{R}[\mathbf{T}]$ *and* $(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}]$ *are theorems.*

Proof. Use reflexivity to show that $\mathbf{R}[\mathbf{T}]$ is a theorem. Build on Metatheorem 93 to show that $(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}]$ is a theorem. □

Metatheorem 96 (part of Bourbaki **C46**). *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Then the following is a theorem:*

$$(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}] \iff (\forall\mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \iff (\mathbf{x} = \tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}]))\Big).$$

Proof. One direction comes from the previous result. So we make the hypotheses $(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}]$. From there the proof is straightforward (using the definition of $\exists$ and Metatheorem 94). □

Remark. Bourbaki discusses how to introduce (in the informal extended language) a "function symbol" Σ associated with a formula $\mathbf{R}[\mathbf{x}]$ where define Σ to be an abbreviation for $\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])$. We could (and should) include parameters for Σ corresponding to the parameters of $\mathbf{R}[\mathbf{x}]$. Adding such a function symbol is especially useful with $(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}]$ is a theorem, so $\mathbf{R}[\Sigma]$ is a theorem and Σ can be thought of as being defined as the unique object satisfying the property described by $\mathbf{R}[\mathbf{x}]$.

We end with a final result proved by Bourbaki. I will call it an exercise since I do not see that it is really essential.

Exercise 4 (Bourbaki **C47**)**.** Let $\mathbf{R}[\mathbf{x}]$ and $\mathbf{S}[\mathbf{x}]$ be formulas with designated variable $\mathbf{x}$. Suppose $(\exists!\mathbf{x})\mathbf{R}[\mathbf{x}]$ is a theorem. Show that the following is a theorem:

$$\mathbf{S}[\tau_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])] \iff (\exists\mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \wedge \mathbf{S}[\mathbf{x}]\Big).$$

9 Formal set theory: first steps. (Bourbaki Ch II, §1)

We have achieved the main purpose of this document which is to consider Bourbaki's Chapter I, covering the formal logic and formal language of Bourbaki's system. However, the beginning of Bourbaki's Chapter II, *Theory of Sets* is closely related to Chapter I, and considers some logical issues, so it seems appropriate to cover the more formal and logical part of Chapter II in this document.

Bourbaki's Chapter II *Theory of Sets* lays the foundation for set theory, and is closely tied to the formal language in the sense that all its results are viewed as translations or abbreviations of formal theorems in the formal language. I plan to present a full commentary of this Chapter II in another document, but this later commentary will approach the theory independently of Bourbaki's particular logical language, and so will describe the theory in a more independent, and hence informal, manner. In this section, on the other hand, we will also cover some of Chapter II, but strictly from Bourbaki's original point of view where it is closely tied to their particular formal language laid out in Chapter I.

In this light, the theory of sets is built on a formal language of the type described in Chapter I. It includes the following specific symbols.

- A binary relational symbol $=$, informally written using infix notation. We define $\neq$ in the usual way in the informal language.
- A binary relational symbol $\in$, informally written using infix notation. We define $\notin$ in the usual way in the informal language.
- A binary functional symbol $\supset$, used to define ordered pairs. In the informal language we will replace $\supset \mathbf{T}\mathbf{U}$ with the ordered pair notation $(\mathbf{T}, \mathbf{U})$, so we will not have much need for this symbol (which should not be confused with the superset symbol).[12]

[12] Bourbaki uses a different font where the distinction between this functional symbol and the superset symbols is clearer. Apparently in later French editions Bourbaki defines ordered pairs in terms of sets as is usual in set theory, and of course there is no need for this functional symbol in such an approach.

The formal language of set theory uses axiom schemes S1 to S7 considered above, together with an extra S8 described below. Five simple axioms A1, A2, A3, A4, A5 are used as well. We will give A1, A2, A3, A4 here, but will leave A5, the axiom of infinity, for another document (it is described and first used in Bourbaki's Chapter III). None of the axioms contain variables, so there are no constants in the basic theory.

This is a pure theory of sets: everything in the universe of discourse is thought of as a set. Elements of all our sets will themselves be sets, and so on. So phrases such as "X is a set" are superfluous, but we can use them for expository purposes.

9.1 Inclusion (Bourbaki Ch II, §1.2)

At first we do not use the new axioms. So our theory has S1 to S7 for now, plus the symbols $=$ and $\in$. This is enough to define the inclusion relation.

Definition 18 (Inclusion)**.** If $\mathbf{T}$ and $\mathbf{U}$ are terms, then we define $\mathbf{T} \subset \mathbf{U}$ to mean

$$(\forall \mathbf{z})(\mathbf{z} \in \mathbf{T} \implies \mathbf{z} \in \mathbf{U})$$

where $\mathbf{z}$ is any variable not in $\mathbf{T}$ or $\mathbf{U}$.

Remark. As usual, we see that the actual formula $(\forall \mathbf{z})(\mathbf{z} \in \mathbf{T} \implies \mathbf{z} \in \mathbf{U})$ does not contain $\mathbf{z}$ (because of how Bourbaki implements bound variables with the $\Box$ symbol and links), and the formula is independent of the choice of $\mathbf{z}$ (see Metatheorem 56, **CS8**)

Remark. Actually Bourbaki adopts new convention for giving definitions using specific expressions instead of a broad metalanguage description. So here, they define $x \subset y$ as

$$(\forall z)((z \in x) \implies (z \in y))$$

with the convention that we can substitute terms $\mathbf{T}$ and $\mathbf{U}$ for x and y respectively. This yields the definition given above. We note that the definition and the above conventions are not of the formal language, but of informal standard mathematical language that will ultimately refers to formal expressions. However, there is a way to make such definitions more formal. For example, starting with theory $\mathcal{T}$ we can form a new theory $\mathcal{T}'$ as follows: we add a binary relation symbol $\subset$, agreeing to use infix notation when using it informally, and then adding the following "definitional stipulation" as a simple axiom

$$(x \subseteq y) \iff ((\forall z)((z \in x) \implies (z \in y))$$

A similar convention applies to definition of terms. If we were to formalize a definition of terms by expanding our theory with a new functional symbol, our "definitional stipulation" would be an axiom with a new axiom with equality $=$ instead of a biconditional $\iff$.

Metatheorem 97 (Bourbaki **CF13**)**.** *If* $\mathbf{T}$ *and* $\mathbf{U}$ *are terms then* $\mathbf{T} \subset \mathbf{U}$ *is a formula.*

Metatheorem 98 (Bourbaki **CS12**)**.** *Suppose* $\mathbf{T}[\mathbf{x}]$ *and* $\mathbf{U}[\mathbf{x}]$ *are terms with designated variable* $\mathbf{x}$*, and* $\mathbf{V}$ *is a term. Then*

$$(\mathbf{V}|\mathbf{x})(\mathbf{T}[\mathbf{x}] \subset \mathbf{U}[\mathbf{x}]) \quad \textit{is the same as} \quad \mathbf{T}[\mathbf{V}] \subset \mathbf{U}[\mathbf{V}].$$

Proof. Apply **CS9** (Metatheorem 57) to $(\forall \mathbf{z})(\mathbf{z} \in \mathbf{T}[\mathbf{x}] \implies \mathbf{z} \in \mathbf{U}[\mathbf{x}])$. □

Using the defintion, the following proposition are fairly straightforward:

Proposition 4 (Bourbaki's Proposition 1)**.** *If* x *is a set then* $x \subset x$*. In other words, we have the theorem*

$$(\forall x)(x \subset x).$$

Remark. As mentioned before assuming x is a set is superfluous since everything is a set. So calling x a set is just an expository device.

Proposition 5 (Bourbaki's Proposition 2)**.** *If* $x \subset y$ *and* $y \subset z$ *then* $x \subset z$*. In other words, we have the theorem*

$$(\forall x)(\forall y)(\forall z)((x \subset y) \wedge (y \subset z) \implies (x \subset z))$$

Remark. In what follows, I will sometimes not mention the universal quantifiers as in the above two propositions. (Actually if the variables in question are not constant, then the version without the quantifiers is logically equivalent to those with the quantifiers.)

9.2 Axiom A1: extensionality (Bourbaki Ch II, §1.3)

Now we are ready to employ some new axioms. The first axiom is A1. Bourbaki calls it the "axiom of extent" but it is more commonly called the "axiom of extensionality" or the "extensionality axiom":

A1: $$(\forall x)(\forall y)\Big((x \subset y) \wedge (y \subset x) \implies (x = y)\Big).$$

We will use this axiom to show that there is at most one set defined by any given formula. Recall we defined the quantifier "there exists at most one" above. Using axiom A1 we get the following:

Metatheorem 99 (Bourbaki's **C48**)**.** *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Let* $\mathbf{y}$ *be a variable distinct from* $\mathbf{x}$ *that does not appear in* $\mathbf{R}[\mathbf{x}]$*. Then there exists at most one* $\mathbf{y}$ *such that*

$$(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)$$

9.3 Collectivizing formulas (Bourbaki Ch II, §1.4)

Definition 19. Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$. The formula $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is the formula

$$(\exists \mathbf{y})(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)$$

where $\mathbf{y}$ is a variable distinct from $\mathbf{x}$ not appearing in $\mathbf{R}[\mathbf{x}]$. In informal terms, $\mathbf{R}[\mathbf{x}]$ is collectivizing if there is a set whose members are exactly the elements satisfying $\mathbf{R}[\mathbf{x}]$. The sentence "$\mathbf{R}[\mathbf{x}]$ is collectivizing in $\mathbf{x}$" is sometimes used to informally represent the formula $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$.

This definition gives us two results:

Metatheorem 100. *Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$. The formula $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is independent of the choice of $\mathbf{y}$ in the above definition. The formula $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ does not contain $\mathbf{x}$ but otherwise has the same variables as $\mathbf{R}[\mathbf{x}]$.*

Remark. The above result follows from basic quantifier laws of $\exists$ and $\forall$. We can also show that $\mathrm{Coll}_{\mathbf{x}}$ acts as a kind of quantifier and satisfies the basic quantifier law (See Section 10.1 and Metatheorem 113).

Metatheorem 101. *Let $\mathbf{R}[\mathbf{x}]$ and $\mathbf{S}[\mathbf{x}]$ be formulas with designated variable $\mathbf{x}$. Then*

$$(\forall \mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \iff \mathbf{S}[\mathbf{x}]\Big) \implies (\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}] \iff \mathrm{Coll}_{\mathbf{x}}\mathbf{S}[\mathbf{x}])$$

is a theorem.

Exercise 5. Show that $x \in y$ is collectivizing in x, but that $x \notin x$ is not. In other words, show that the following are theorem: $\mathrm{Coll}_x(x \in y)$ and $\neg\mathrm{Coll}_x(x \notin x)$. In particular, not all formulas are collectivizing. (Hint: use the strategy of Russell's paradox.)

Metatheorem 102 (Bourbaki's **C49**)**.** *Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$. Then $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is logically equivalent to the formula*

$$(\exists! \mathbf{y})(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)$$

where $\mathbf{y}$ is a variable distinct from $\mathbf{x}$ not appearing in $\mathbf{R}[\mathbf{x}]$.

Proof. We break the proof of logical equivalence into two directions. One direction follows from the definition of unique existence $\exists!$ and the definition of $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$. For the other direction, assume $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$. In other words, assume $(\exists \mathbf{y})\mathbf{S}[\mathbf{y}]$ where $\mathbf{S}[\mathbf{y}]$ is the formula $(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)$. Let $\mathbf{z}, \mathbf{w}$ be distinct (nonconstant) variables different from $\mathbf{x}$ and any other variable found in $\mathbf{R}[\mathbf{x}]$. Note that by **CS9** (Metatheorem 57) to get

$$\mathbf{S}[\mathbf{z}] \implies (\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{z}) \iff \mathbf{R}[\mathbf{x}]\Big) \quad \text{and} \quad \mathbf{S}[\mathbf{w}] \implies (\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}[\mathbf{x}]\Big).$$

From this it is straightforward to prove, from the assumption $\mathbf{S}[\mathbf{z}] \wedge \mathbf{S}[\mathbf{z}]$, that $\mathbf{z} \subset \mathbf{w}$ and $\mathbf{w} \subset \mathbf{z}$, and so $\mathbf{z} = \mathbf{w}$ from Axiom A1. Thus

$$(\forall \mathbf{z})(\forall \mathbf{w})\Big(\mathbf{S}[\mathbf{z}] \wedge \mathbf{S}[\mathbf{w}] \implies \mathbf{z} = \mathbf{w}\Big).$$

In other words, there is at most one $\mathbf{x}$ such that $\mathbf{S}[\mathbf{x}]$. So we get $(\exists! \mathbf{x})\mathbf{S}[\mathbf{x}]$ as desired. □

Definition 20. Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$. Then

$$\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$$

is defined to be $\tau_{\mathbf{y}}\Big((\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)\Big)$ where we assume $\mathbf{y}$ is not $\mathbf{x}$ and does not appear in $\mathbf{R}[\mathbf{x}]$. Note: this formula is independent of the choice of $\mathbf{y}$ (see **CS3**, Metatheorem 7), and $\mathbf{x}$ does not appear in $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$. So we regard $\mathbf{x}$ as a bound variable for this notation. Otherwise $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$ has the same variables appearing as $\mathbf{R}[\mathbf{x}]$; in other words its variables are equal to the parameters of $\mathbf{R}[\mathbf{x}]$.

Remark. In practice, this definition is used only when $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is a theorem. Bourbaki uses the notation $\mathcal{E}_{\mathbf{x}}(\mathbf{R}[\mathbf{x}])$ instead of $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$, but $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$ is much more standard.

As mentioned above, $\mathbf{x}$ is a bound variable in this notation. We can make this more precise (see Secton 10.1, including Metatheorem 114). Such rules include the following:

Metatheorem 103. *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. If* $\mathbf{x}'$ *is not a parameter of* $\mathbf{R}[\mathbf{x}]$ *then* $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$ *is identical, as a term, to* $\{\mathbf{x}' \mid \mathbf{R}[\mathbf{x}']\}$*.*

As mentioned above, we usually use the notation $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$ when $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is a theorem. In this situation we can take advantage of the following:

Metatheorem 104. *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Then* $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ *is identical to the formula*

$$(\forall \mathbf{x})\Big((\mathbf{x} \in \{\mathbf{z} \mid \mathbf{R}[\mathbf{z}]\}) \iff \mathbf{R}[\mathbf{x}]\Big)$$

where $\mathbf{z}$ *is any variable not equal to a parameter of* $\mathbf{R}[\mathbf{x}]$*.*

Proof. Let $\mathbf{y}$ is any variable not equal to $\mathbf{x}$ or a parameter of $\mathbf{R}[\mathbf{x}]$. By definition, the formula $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ is just

$$(\exists \mathbf{y})(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big).$$

By the definition of the existential quantifier (Definition 13), this formula is just

$$\big(\mathbf{T}|\mathbf{y}\big)(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)$$

where $\mathbf{T}$ is $\tau_{\mathbf{y}}\Big((\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{y}) \iff \mathbf{R}[\mathbf{x}]\Big)\Big)$. By rules of quantifiers for $\forall \mathbf{x}$ (see **CS9**, Metatheorem 57, or more generally Section 10.1) our formula is

$$(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{T}) \iff \mathbf{R}[\mathbf{x}]\Big).$$

Note that $\mathbf{T}$ is just $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$, by definition. For convenience, we can rewrite $\mathbf{T}$ as $\{\mathbf{z} \mid \mathbf{R}[\mathbf{z}]\}$ (by the previous result), and the result follows. □

A very interesting feature of Bourbaki's notational conventions is that the above states that two formulas are *identical*, not just logically equivalent. From the above it is straightforward to derive the following:

Metatheorem 105 (Bourbaki's **C50**). *Let* $\mathbf{R}[\mathbf{x}]$ *and* $\mathbf{S}[\mathbf{x}]$ *be formulas with designated variable* $\mathbf{x}$ *such that* $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ *and* $\mathrm{Coll}_{\mathbf{x}}\mathbf{S}[\mathbf{x}]$ *are theorems. Then*

$$(\forall \mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \implies \mathbf{S}[\mathbf{x}]\Big) \iff \{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\} \subseteq \{\mathbf{x} \mid \mathbf{S}[\mathbf{x}]\}$$

and

$$(\forall \mathbf{x})\Big(\mathbf{R}[\mathbf{x}] \iff \mathbf{S}[\mathbf{x}]\Big) \iff \{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\} = \{\mathbf{x} \mid \mathbf{S}[\mathbf{x}]\}$$

are theorems.

9.4 Axioms A2, A3, A4 (Some of Bourbaki Ch II, §1.5, §2.1, and §5.1)

In addition to the extensionality axiom A1 considered above, Bourbaki's set theory features three more simple axioms (besides the axiom of infinity, which we do not consider here). There is also a critically important axiom scheme which we discuss in the next section. (Note the use of x, y and so on for variables, not syntactic variables such as $\mathbf{x}, \mathbf{y}$. So x, y and so on designated specific variables.)

The *pairing axiom* is

A2. $\quad (\forall x)(\forall y)\mathrm{Coll}_z\Big((z = x) \vee (z = y)\Big).$

The *ordered pair axiom* is

A3. $\quad (\forall x)(\forall y)(\forall x')(\forall y')\Big((\supset xy) = (\supset x'y') \implies (x = x') \wedge (y = y')\Big).$

Informally we write $\supset xy$ using ordered pair notation (x, y). Then A3 can be written as

A3. $\quad (\forall x)(\forall y)(\forall x')(\forall y')\Big((x, y) = (x', y') \implies (x = x') \wedge (y = y')\Big).$

The *power set axiom* is

A4. $\quad (\forall X)\ \mathrm{Coll}_Y\Big(Y \subset X\Big).$

Definition 21. Given x and y, we define $\{x, y\}$ to be $\{z \mid (z = x) \vee (z = y)\}$.

Remark. Note that this definition uses the convention of Bourbaki that a definition with specific variables applies to general terms. In other words, the definition means that

$$(\forall x)(\forall y)\Big(\{x, y\} = \{z \mid (z = x) \vee (z = y)\}\Big)$$

is a theorem, and we can specialize to specific terms (using laws of quantifiers, see Section 10.1). Whether we regard this as implemented as an abbreviation, or via the introduction of a new function symbol, we have such metatheoretical results as the identity of $(\mathbf{U}|\mathbf{x})\{\mathbf{S}, \mathbf{T}\}$ with $\{(\mathbf{U}|\mathbf{x})\mathbf{S}, (\mathbf{U}|\mathbf{x})\mathbf{T}\}$.

Proposition 6. *For all* x, y, z*,*

$$z \in \{x, y\} \iff (z = x) \vee (z = y).$$

Proof. See A2 and Metatheorem 104. □

Proposition 7. *For all* x, y*,*

$$\{x, y\} = \{y, x\}.$$

Definition 22. Given x, we define $\{x\}$ to be $\{x, x\}$.

Proposition 8. *For all* x, z*,*

$$z \in \{x\} \iff z = x.$$

Proposition 9. *For all* x, X*,*

$$x \in X \iff \{x\} \subset X.$$

Definition 23. We define $\mathfrak{P}(X)$ to be $\{Y \mid Y \subset X\}$ which is

$$\tau_Z(\forall Y)(Y \in Z \iff Y \subset X).$$

By Metatheorem 104, Axiom A4 is the same as the following formula

$$(\forall X)(\forall Y)\Big(Y \in \mathfrak{P}(X) \iff Y \subset X\Big).$$

Thus

Proposition 10. *For all* X, Y*,*

$$Y \in \mathfrak{P}(X) \iff Y \subset X.$$

Proposition 11. *For all* X, X'*, if* $X \subset X'$ *then* $\mathfrak{P}(X) \subset \mathfrak{P}(X')$*.*

9.5 The set theory scheme (Bourbaki Ch 2, §1.6)

There are seven axiom schemes among Bourbaki's logical axioms. Bourbaki introduces only one extra axiom scheme in the theory of sets called *the scheme of selection and union* since it combines the traditional separation and union axioms of Zermelo-Fraenkel set theory. It also implies the traditional replacement axiom. Informally it says the following: suppose $\mathbf{R}[x, y]$ is a relation (defined by a formula, perhaps with extra parameters) such that for each y there is a set X_y capturing all the x with $\mathbf{R}[x, y]$. Then for all sets Y, we have the existence of a set

$$\{x \mid \mathbf{R}[x, y] \text{ for some } y \in Y\}.$$

Here is the precise version:

S8. Suppose $\mathbf{R}$ is a formula and let $\mathbf{x}$ and $\mathbf{y}$ be distinct variables. Let $\mathbf{X}$ and $\mathbf{Y}$ be variables distinct from $\mathbf{x}, \mathbf{y}$ and any variable in $\mathbf{R}$. Then

$$\Big((\forall \mathbf{y})(\exists \mathbf{X})(\forall \mathbf{x})\big(\mathbf{R} \implies (\mathbf{x} \in \mathbf{X})\big)\Big) \implies (\forall \mathbf{Y})\mathrm{Coll}_{\mathbf{x}}\big((\exists \mathbf{y})((\mathbf{y} \in \mathbf{Y}) \wedge \mathbf{R})\big)$$

is an axiom.

Metatheorem 106. *The above scheme* S8 *is an axiom scheme.*

Proof. We must show that S8 is closed under substitution of a term $\mathbf{T}$ for a variable $\mathbf{u}$. Given an instance $\mathbf{B}$ of S8, we can use the laws of quantifiers (Section 10.1) to replace the bound variables with variables not occurring in $\mathbf{T}$ and not equal to $\mathbf{u}$. Now we use the laws of quantifiers (Section 10.1) to write $(\mathbf{T}|\mathbf{u})\mathbf{B}$ in the form given by S8. □

This axiom scheme can be used to justify the separation property that states that given a set A and a property $P(x)$, there is a subset of A consisting of elements satisfying the property $P(x)$. This actually an axiom in traditional Zermelo-Fraenkel set theory.

Metatheorem 107 (Separation Property, Bourbaki's **C51**)**.** *Suppose* $\mathbf{P}[\mathbf{x}]$ *is a formula with designated variable* $\mathbf{x}$*. Suppose* $\mathbf{A}$ *is a term and that* $\mathbf{x}$ *does not appear in* $\mathbf{A}$*. Then*

$$\mathrm{Coll}_{\mathbf{x}}\Big(\mathbf{P}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A})\Big)$$

is a theorem.

Proof. (This theorem is in a theory without constants, so we can assume $\mathbf{x}$ is not constant. Even if we were in a theory with constants, we could change the bound variable $\mathbf{x}$ if necessary to not be constant.) Let $\mathbf{y}$ and $\mathbf{X}$ be distinct variable that are each distinct from $\mathbf{x}$ and any variable of $\mathbf{P}[\mathbf{x}]$ or $\mathbf{A}$ (and distinct from any constant). Let $\mathbf{R}[\mathbf{x}, \mathbf{y}]$ be the formula $\mathbf{P}[\mathbf{x}] \wedge (\mathbf{x} = \mathbf{y})$. Observe that $\mathbf{R}[\mathbf{x}, \mathbf{y}]$ implies that $\mathbf{x} \in \{\mathbf{y}\}$ (Proposition 8). Thus we have

$$(\forall \mathbf{y})(\exists \mathbf{X})(\forall \mathbf{x})\big(\mathbf{R}[\mathbf{x}, \mathbf{y}] \implies (\mathbf{x} \in \mathbf{X})\big).$$

By S8 (and the standard substitution and quantifier laws, see Section 10.1, to specialize from $\mathbf{Y}$ to $\mathbf{A}$) we get

$$\mathrm{Coll}_{\mathbf{x}}\big((\exists \mathbf{y})((\mathbf{y} \in \mathbf{A}) \wedge \mathbf{R}[\mathbf{x}, \mathbf{y}])\big).$$

The result now follows from Metatheorem 101. □

Definition 24. We write

$$\{\mathbf{x} \in \mathbf{A} \mid \mathbf{P}[\mathbf{x}]\}$$

for

$$\{\mathbf{x} \mid \mathbf{P}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A})\}.$$

By Metatheorem 104, the formula $\mathrm{Coll}_{\mathbf{x}}\Big(\mathbf{P}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A})\Big)$ is the same as the following formula

$$(\forall \mathbf{x})\Big(\mathbf{x} \in \{\mathbf{w} \in \mathbf{A} \mid \mathbf{P}[\mathbf{w}]\} \iff \mathbf{P}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A})\Big)$$

where $\mathbf{w}$ is any variable not appearing in $\mathbf{A}$ or as a parameter of $\mathbf{P}[\mathbf{x}]$. So we get the following:

Metatheorem 108. *Let* $\mathbf{P}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$*. Suppose* $\mathbf{A}$ *is a term without the variable* $\mathbf{x}$*, and let* $\mathbf{T}$ *be any term. Then* $\mathbf{T} \in \{\mathbf{x} \in \mathbf{A} \mid \mathbf{P}[\mathbf{x}]\}$ *if and only if* $\mathbf{P}[\mathbf{T}]$ *and* $\mathbf{T} \in \mathbf{A}$*.*

Metatheorem 109 (Bourbaki's **C52**)**.** *Let* $\mathbf{R}[\mathbf{x}]$ *be a formula with designated variable* $\mathbf{x}$ *that is not a constant, and let* $\mathbf{A}$ *a term that does not contain* $\mathbf{x}$*. If* $\mathbf{R}[\mathbf{x}] \implies \mathbf{x} \in \mathbf{A}$ *is a theorem, then* $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}$ *is also a theorem.*

Proof. Observe that $(\forall \mathbf{x})\big((\mathbf{R}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A}) \iff \mathbf{R}[\mathbf{x}]\big)$ is a theorem. So by Metatheorem 101

$$\mathrm{Coll}_{\mathbf{x}}\left(\mathbf{R}[\mathbf{x}] \wedge (\mathbf{x} \in \mathbf{A})\right) \iff \mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}].$$

The result now follows from **C51** (Metatheorem 107, the separation property). □

Corollary. *Suppose* $\mathbf{R}[\mathbf{x}]$ *and* $\mathbf{S}[\mathbf{x}]$ *are formulas with designated variable* $\mathbf{x}$*. If* $\mathrm{Coll}_{\mathbf{x}}\mathbf{R}[\mathbf{x}]$ *and* $(\forall \mathbf{x})(\mathbf{S}[\mathbf{x}] \implies \mathbf{R}[\mathbf{x}])$ *then* $\mathrm{Coll}_{\mathbf{x}}\mathbf{S}[\mathbf{x}]$ *and*

$$\{\mathbf{x} \mid \mathbf{S}[\mathbf{x}]\} \subset \{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$$

Proof. Let $\mathbf{A}$ be $\{\mathbf{x} \mid \mathbf{R}[\mathbf{x}]\}$. Then by Metatheorem **??**,

$$(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{A}) \iff \mathbf{R}[\mathbf{x}]\Big).$$

Changing $\mathbf{x}$ if necessary, we can assume $\mathbf{x}$ is not a constant in the above. Observe that now we have $\mathbf{S}[\mathbf{x}] \implies (\mathbf{x} \in \mathbf{A})$ so $\mathrm{Coll}_{\mathbf{x}}\mathbf{S}[\mathbf{x}]$ by the above result. Again by Metatheorem **??**, if $\mathbf{B}$ is $\{\mathbf{x} \mid \mathbf{S}[\mathbf{x}]\}$ then

$$(\forall \mathbf{x})\Big((\mathbf{x} \in \mathbf{B}) \iff \mathbf{S}[\mathbf{x}]\Big).$$

Thus $(\forall \mathbf{x})((\mathbf{x} \in \mathbf{B}) \implies (\mathbf{x} \in \mathbf{A}))$, so $\mathbf{B} \subset \mathbf{A}$. □

Besides the separation axiom, axiom scheme S8 also gives the replacement axiom of the more traditional Zermelo-Fraenkel version of set theory. This principle implies that the image of a set under an operation definable in the formal language. In particular, given a set A and an operation $x \mapsto T[x]$ the image $\{T[a] \mid a \in A\}$ is a set. The following makes this precise:

Metatheorem 110 (Replacement principle, Bourbaki's **C53**)**.** *Suppose that* $\mathbf{T}$ *is a term with designated variable* $\mathbf{x}$*. Suppose* $\mathbf{A}$ *is a term which does not contain* $\mathbf{x}$*. Then*

$$\mathrm{Coll}_{\mathbf{y}}(\exists \mathbf{x})((\mathbf{y} = \mathbf{T}[\mathbf{x}]) \wedge (\mathbf{x} \in \mathbf{A}))$$

where $\mathbf{y}$ *is a variable distinct from* $\mathbf{x}$ *not appearing in* $\mathbf{T}[\mathbf{x}]$ *or* $\mathbf{A}$*.*

Proof. By changing bound variables $\mathbf{x}$ and $\mathbf{y}$ if necessary, we can write the desired formula in terms of $\mathbf{x}$ and $\mathbf{y}$ that are not constants (or just stick to the base theory with no constants). We represent $\mathbf{T}[\mathbf{x}]$ simply as $\mathbf{T}$, and we start with the following axiom produced by axiom scheme S8.

$$(\forall \mathbf{x})(\exists \mathbf{X})(\forall \mathbf{y})\big((\mathbf{y} = \mathbf{T}) \implies (\mathbf{y} \in \mathbf{X})\big) \implies (\forall \mathbf{Y})\mathrm{Coll}_{\mathbf{y}}(\exists \mathbf{x})((\mathbf{x} \in \mathbf{Y}) \wedge (\mathbf{y} = \mathbf{T}))$$

where $\mathbf{X}$ and $\mathbf{Y}$ are variables chosen to be distinct from $\mathbf{x}$ and $\mathbf{y}$ and any other variable in $\mathbf{T}$. By Proposition 8 we have

$$(\forall \mathbf{y})((\mathbf{y} = \mathbf{T}) \implies (\mathbf{y} \in \{\mathbf{T}\}))$$

and so by $\exists$-introduction and $\forall$-introduction we get

$$(\forall \mathbf{x})(\exists \mathbf{X})(\forall \mathbf{y})\big((\mathbf{y} = \mathbf{T}) \implies (\mathbf{y} \in \mathbf{X})\big)$$

and so

$$(\forall \mathbf{Y})\mathrm{Coll}_{\mathbf{y}}(\exists \mathbf{x})((\mathbf{x} \in \mathbf{Y}) \wedge (\mathbf{y} = \mathbf{T})).$$

By $\forall$-elimination, we can replace $\mathbf{Y}$ by $\mathbf{A}$, and since $\mathbf{x}$ and $\mathbf{y}$ do not appear in $\mathbf{A}$ (and $\mathbf{Y}$ does not appear in $\mathbf{T}$) we get

$$\mathrm{Coll}_{\mathbf{y}}(\exists \mathbf{x})((\mathbf{x} \in \mathbf{A}) \wedge (\mathbf{y} = \mathbf{T})).$$

We use Metatheorem 101 gives the desired result by allowing us to switch the order of the terms around the $\wedge$. □

Finally we consider the existence of unions, which is actually an axiom in Zermelo-Fraenkel set theory. Bourbaki delay's their treatment of Unions to their section §4.1, and their treatment is in terms of the union of a family of sets. However, in the spirit of commentary, it is worth seeing how the axiom of union of Zermelo-Fraenkel set theory falls out of axiom scheme S8. This helps to explain why Bourbaki calls S8 the scheme of selection and union.

Definition 25. Let Y be a set, which we think of as a collection of sets. Then we define the union of Y, written $\bigcup Y$ by the following equation:

$$\bigcup Y = \{x \mid (\exists y)((y \in Y) \wedge (x \in y)\}$$

(Note: we are adopting Bourbaki's convention for definition by using a specific equation with specific variables, which is asserted to hold universally upon replacement of Y by an arbitrary term $\mathbf{T}$ via substitution $(\mathbf{T}|Y)$.)

Theorem 12 (Existence of the union)**.** *For all Y,*

$$\mathrm{Coll}_x(\exists y)((y \in Y) \wedge (x \in y)).$$

In particular, for all Y and x

$$\Big(x \in \bigcup Y\Big) \iff (\exists y)((y \in Y) \wedge (x \in y)).$$

Proof. From Axiom Scheme S8 we generate the following specific theorem:

$$(\forall y)(\exists X)(\forall x)\big((x \in y) \implies (x \in X)\big) \implies (\forall Y)\mathrm{Coll}_x(\exists y)((y \in Y) \wedge (x \in y)).$$

From the theorem $(\forall x)\big((x \in y) \implies (x \in y)\big)$ and $\exists$-introduction we get

$$(\exists X)(\forall x)\big((x \in y) \implies (x \in X)\big).$$

We assume y is not a constant in our current theory (if it is then replace y in the above with a true variable, and switch back to y later). Then, by $\forall$-introduction, we get

$$(\forall y)(\exists X)(\forall x)\big((x \in y) \implies (x \in X)\big)$$

and so

$$(\forall Y)\mathrm{Coll}_x(\exists y)((y \in Y) \wedge (x \in y))$$

as desired. The second claim now follows from Metatheorem 104 and the definition of union. □

10 Further commentary

In this section we consider various issues related to the above. This section may be expanded in later editions of this document.

10.1 Laws of quantifiers and bounded variables

Let's define a *quantifier* as a family of operators, one for each variable, mapping a class of expressions to a class of expressions, that satisfies standard properties set out below. (There might be more properties worth highlighting, but these give most of what we need). The specific notation will vary, but suppose we have a quantifier Q where we denote $Q_{\mathbf{x}}$ the operator associated to the variable $\mathbf{x}$. If $\mathbf{A}$ is an expression in the domain of $Q_{\mathbf{x}}$ then we denote by $Q_{\mathbf{x}}\mathbf{A}$ or $Q_{\mathbf{x}}(\mathbf{A})$ the image of $\mathbf{A}$ under the operator. Here are some required properties:

1. For any expression $\mathbf{A}$ in the domain class of Q, the variable $\mathbf{x}$ should not occur in the expression $Q_{\mathbf{x}}\mathbf{A}$.[13]

2. Aside from $\mathbf{x}$, a variable occurs in $Q_{\mathbf{x}}\mathbf{A}$ if and only if it occurs in $\mathbf{A}$.

3. We can change variables as follows: if $\mathbf{y}$ does not occur in $\mathbf{A}$ then

   $$Q_{\mathbf{x}}\mathbf{A} \quad \text{is the same as} \quad Q_{\mathbf{y}}(\mathbf{y}|\mathbf{x})\mathbf{A}.$$

   This also holds if $\mathbf{y}$ is $\mathbf{x}$ since $(\mathbf{x}|\mathbf{x})$ is the identity operator.

4. Under reasonable circumstances, substitution commutes with $Q_{\mathbf{x}}$. More specifically, if $\mathbf{y}$ is distinct from $\mathbf{x}$, and if $\mathbf{T}$ is a term that does not contain $\mathbf{x}$, then the substitution operator $(\mathbf{T}|\mathbf{y})$ commutes with $Q_{\mathbf{x}}$.

[13] In Bourbaki's system we eliminate $\mathbf{x}$ by replacement with the symbol □. In other systems $\mathbf{x}$ might occur as a bound variable, but $Q_{\mathbf{x}}A$ should have no free instances of $\mathbf{x}$.

There are several examples of quantifiers in Bourbaki's first two chapters:

- The τ-operator $\tau_{\mathbf{x}}$ is a quantifier in the above sense, mapping formulas to terms. See Metatheorem 1, Metatheorem 7 (**CS3**), and Metatheorem 8 (**CS4**).
- The existential quantifier $(\exists \mathbf{x})$ is a quantifier in the above sense mapping formulas to formulas. See Metatheorem 55 (**CF11**), Metatheorem 56 (**CS8**), and Metatheorem 57 (**CS9**).
- The universal quantifier $(\forall \mathbf{x})$ is a quantifier in the above sense mapping formulas to formulas. See Metatheorem 55 (**CF11**), Metatheorem 56 (**CS8**), and Metatheorem 57 (**CS9**).
- The operators $(\mathbf{A}, \mathbf{R}) \mapsto (\exists_{\mathbf{A}} \mathbf{x})\mathbf{R}$ and $(\mathbf{A}, \mathbf{R}) \mapsto (\forall_{\mathbf{A}} \mathbf{x})\mathbf{R}$ taking pairs of formulas to formulas are almost quantifiers in the above sense, but we need to consider two inputs $\mathbf{A}$ and $\mathbf{R}$ instead of one in the quantifier laws. See Metatheorem 74, Metatheorem 75 (**CS10**), Metatheorem 76 (**CS11**).
- If $\mathbf{R}$ is a formula, then write $(!\mathbf{x})\mathbf{R}$ to indicate "there exists at most one $\mathbf{x}$ such that $\mathbf{R}$" (see Definition 16). Then $(!\mathbf{x})$ is a quantifier in the above sense mapping formulas to formulas. See the following metatheorem (Metatheorem 111).
- The unique existential quantifier $(\exists ! \mathbf{x})$ is a quantifier in the above sense mapping formulas to formulas. See the following metatheorem (Metatheorem 112).
- The operator $\mathrm{Coll}_{\mathbf{x}}$ (see Definition 19) is a quantifier in the above sense mapping formulas to formulas. See Metatheorem 113 below.
- Consider the set building operator $\mathcal{E}_{\mathbf{x}}$ that maps a formula $\mathbf{R}$ to the term $\{\mathbf{x} \mid \mathbf{R}\}$ (see Definition 20). This is a quantifier in the above sense mapping formulas to terms. See Metatheorem 114 below.

Metatheorem 111. *The operator* $(!\mathbf{x})$ *satisfies the quantifier laws.*

Proof. Let $\mathbf{R}[\mathbf{x}]$ be a formula with designated variable $\mathbf{x}$, which we can write simply as $\mathbf{R}$. The formula $(!\mathbf{x})\mathbf{R}$ has the desired variables according to Metatheorem 92, so there are only two remaining laws to check. By Definition 16 and the definition of a quantifier, the remaining items to check are as follows:

1. Choose $\mathbf{z}, \mathbf{w}$ to be distinct variables that are not parameters of $\mathbf{R}[\mathbf{x}]$, and let $\mathbf{y}$ be a variable not in $\mathbf{R}$. Then we must check that

$$(\forall \mathbf{z})(\forall \mathbf{w})\Big((\mathbf{z}|\mathbf{y})(\mathbf{y}|\mathbf{x})\mathbf{R} \wedge (\mathbf{w}|\mathbf{y})(\mathbf{y}|\mathbf{x})\mathbf{R} \implies \mathbf{z} = \mathbf{w}\Big)$$

   is the same as

$$(\forall \mathbf{z})(\forall \mathbf{w})\Big((\mathbf{z}|\mathbf{x})\mathbf{R} \wedge (\mathbf{w}|\mathbf{x})\mathbf{R} \implies \mathbf{z} = \mathbf{w}\Big).$$

   This follows from **CS1** (Metatheorem 5) since $\mathbf{y}$ is not in $\mathbf{R}$.

2. Let $\mathbf{T}$ be a term not containing $\mathbf{x}$, and let $\mathbf{y}$ be a variable distinct from $\mathbf{x}$. Choose $\mathbf{z}$ and $\mathbf{w}$ to distinct variables that are not parameters of $\mathbf{R}[\mathbf{x}]$, and that are distinct from $\mathbf{y}$ and any variable in $\mathbf{T}$. Then we must check that

$$(\mathbf{T}|\mathbf{y})(\forall\mathbf{z})(\forall\mathbf{w})\Big((\mathbf{z}|\mathbf{x})\mathbf{R} \wedge (\mathbf{w}|\mathbf{x})\mathbf{R} \implies \mathbf{z} = \mathbf{w}\Big)$$

is the same as

$$(\forall\mathbf{z})(\forall\mathbf{w})\Big((\mathbf{z}|\mathbf{x})(\mathbf{T}|\mathbf{y})\mathbf{R} \wedge (\mathbf{w}|\mathbf{x})(\mathbf{T}|\mathbf{y})\mathbf{R} \implies \mathbf{z} = \mathbf{w}\Big).$$

This follows from the quantifier laws of $\forall$, and laws about the substitution operator $(\mathbf{T}|\mathbf{y})$ including **CS2** (Metatheorem 6) which implies that $(\mathbf{T}|\mathbf{y})$ commutes with $(\mathbf{z}|\mathbf{x})$ and $(\mathbf{w}|\mathbf{x})$ since $\mathbf{x}$ does not appear in $\mathbf{T}$.

□

Metatheorem 112. *The operator* $(\exists!\mathbf{x})$ *satisfies the quantifier laws.*

Proof. This follows from the fact that $(!\mathbf{x})$ and $\exists\mathbf{x}$ individually satisfy the quantifier laws and that $(\exists!\mathbf{x})\mathbf{R}$ is just $((\exists\mathbf{x})\mathbf{R}) \wedge ((!\mathbf{x})\mathbf{R})$. □

Metatheorem 113. *The operator* $\text{Coll}_{\mathbf{x}}$ *satisfies the quantifier laws.*

Proof. Let $\mathbf{R}$ be a formula and $\mathbf{x}$ be a variable. The formula $\text{Coll}_{\mathbf{x}}\mathbf{R}$ has the desired variables according to Metatheorem 100, so there are only two remaining laws to check. By Definition 19 and the definition of a quantifier, the remaining items to check are as follows:

1. Let $\mathbf{y}$ be a variable distinct from $\mathbf{x}$ that does not appear in $\mathbf{R}$. Choose $\mathbf{w}$ to be a variable distinct from $\mathbf{x}$ and $\mathbf{y}$ and any variable of $\mathbf{R}$. Then we must check that

$$(\exists\mathbf{w})(\forall\mathbf{y})\Big((\mathbf{y} \in \mathbf{w}) \iff (\mathbf{y}|\mathbf{x})\mathbf{R}\Big)$$

is the same as

$$(\exists\mathbf{w})(\forall\mathbf{x})\Big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\Big).$$

To see this observe that both formulas are the same as

$$(\exists\mathbf{w})(\forall\mathbf{y})(\mathbf{y}|\mathbf{x})\Big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\Big)$$

(where we use the laws of quantifiers for $\forall$).

2. Let $\mathbf{T}$ be a term not containing $\mathbf{x}$, and let $\mathbf{y}$ be a variable distinct from $\mathbf{x}$. Choose $\mathbf{w}$ to be a variable distinct from $\mathbf{x}$ and $\mathbf{y}$ and any variable of $\mathbf{R}$ or $\mathbf{T}$. Then we must check that

$$(\mathbf{T}|\mathbf{y})(\exists\mathbf{w})(\forall\mathbf{x})\Big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\Big)$$

is the same as

$$(\exists\mathbf{w})(\forall\mathbf{x})\Big((\mathbf{x} \in \mathbf{w}) \iff (\mathbf{T}|\mathbf{y})\mathbf{R}\Big).$$

This follows from the quantifier laws of $\exists$ and $\forall$.

□

Metatheorem 114. *The set building operator $\mathcal{E}_{\mathbf{x}}$ satisfies the quantifier laws.*

Proof. Throughout let $\mathbf{R}$ be a formula and $\mathbf{x}$ be a variable. By Definition 20, $\mathcal{E}_{\mathbf{x}}\mathbf{R}$ is $\tau_{\mathbf{w}}(\forall\mathbf{x})\big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\big)$ where we assume $\mathbf{w}$ is not $\mathbf{x}$ and does not appear in $\mathbf{R}$. By the quantifier laws for $\tau_{\mathbf{w}}$ and $\forall\mathbf{x}$, the variable $\mathbf{x}$ does not appear in $\mathcal{E}_{\mathbf{x}}\mathbf{R}$ but otherwise $\mathcal{E}_{\mathbf{x}}\mathbf{R}$ and $\mathbf{R}$ have the same variables. Also the quantifier laws for $\tau_{\mathbf{w}}$ and $\forall\mathbf{x}$ can be used to show that the resulting term does not depend on the particular choice of $\mathbf{w}$.

So there are only two remaining laws to check. By Definition 20 and the definition of a quantifier, the remaining items to check are as follows:

1. Let $\mathbf{y}$ be a variable distinct from $\mathbf{x}$ that does not appear in $\mathbf{R}$. Choose $\mathbf{w}$ to be a variable distinct from $\mathbf{x}$ and $\mathbf{y}$ and any variable of $\mathbf{R}$. Then we must check that
$$\tau_{\mathbf{w}}(\forall\mathbf{y})\big((\mathbf{y} \in \mathbf{w}) \iff (\mathbf{y}|\mathbf{x})\mathbf{R}\big)$$
is the same as
$$\tau_{\mathbf{w}}(\forall\mathbf{x})\big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\big)$$
To see this observe that both formulas are the same as
$$\tau_{\mathbf{w}}(\forall\mathbf{y})(\mathbf{y}|\mathbf{x})\big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\big)$$
(where we use the laws of quantifiers for $\forall$).

2. Let $\mathbf{T}$ be a term not containing $\mathbf{x}$, and let $\mathbf{y}$ be a variable distinct from $\mathbf{x}$. Choose $\mathbf{w}$ to be a variable distinct from $\mathbf{x}$ and $\mathbf{y}$ and any variable of $\mathbf{R}$ or $\mathbf{T}$. Then we must check that
$$(\mathbf{T}|\mathbf{y})\tau_{\mathbf{w}}(\forall\mathbf{x})\big((\mathbf{x} \in \mathbf{w}) \iff \mathbf{R}\big)$$
is the same as
$$\tau_{\mathbf{w}}(\forall\mathbf{x})\big((\mathbf{x} \in \mathbf{w}) \iff (\mathbf{T}|\mathbf{y})\mathbf{R}\big)$$
This follows from the established quantifier laws of $\tau_{\mathbf{w}}$ and $\forall\mathbf{x}$.

□