Definition. If \( f : V \to V' \) is a linear map between vector spaces (or modules), then the kernel of \( f \) or the null-space of \( f \) is the set \( \{ v \in V \mid f(v) = 0 \} \).

Problems 1–4: Null-spaces (or kernels). Let \( f : V \to V' \) be a linear map between vector spaces (or modules).
1. Show that the kernel of \( f \) is a subspace of \( V \).
2. Show \( f \) is injective (one-to-one) if and only if the kernel of \( f \) is equal to the zero subspace.
3. Show that if \( u_1, \ldots, u_n \) is a linearly independent sequence of vector of \( V_1 \), and if \( f \) is injective, then \( f(u_1), \ldots, f(u_n) \) is a linearly independent sequence of vectors of \( V_2 \). (Optional: does this work for infinite families?)
4. Give a counter-example to the above when \( f \) is not injective. (Hint look at \( F^3 \to F^2 \) defined by the rule \((x_1, x_2, x_3) \mapsto (x_1, x_2))\).

Definition. If \( f : V \to V' \) is a linear map between vector spaces (or modules), then the image of \( f \) or the range of \( f \) is the set \( W \) of vectors in \( V' \) of the form \( f(v) \) with \( v \in V \).

Problems 5–7: Image spaces (or ranges). Let \( f : V \to V' \) be a linear map between vector spaces (or modules).
5. Show that the image of \( f \) is a subspace of \( V' \). Observe that \( f \) is surjective (onto) if and only if the image of \( f \) is equal to all of \( V' \).
6. Show that if \( u_1, \ldots, u_n \) span \( V_1 \), and if \( f \) is surjective, then \( f(u_1), \ldots, f(u_n) \) spans \( V_2 \). (Optional: does this work for infinite spanning sets?)
7. Give a counter-example to the above when \( f \) is not surjective. (Hint look at \( F^2 \to F^3 \) defined by the rule \((x_1, x_2) \mapsto (x_1, x_2, 0))\).

Problems 8–11: Isomorphisms. In mathematics, an isomorphism is a homomorphism that has an inverse homomorphism. In algebra, this is usually equivalent to requiring that the homomorphism be bijective (one-to-one and onto). In topology, bijective is not enough. We will show that in linear algebra it is enough.

Definition. If \( f : A \to B \) and \( g : B \to A \) are functions such that \( f \circ g = \text{id}_B \) and \( g \circ f = \text{id}_A \), then we say that \( f \) and \( g \) are inverse functions.

8. (Review?) Show that a function \( f \) has an inverse function if and only if \( f \) is injective (one-to-one) and surjective (onto). Functions that are both injective and surjective are bijective.

Definition. If \( f : V \to V' \) is a homomorphism that has an inverse function \( f^{-1} : V' \to V \), and if the inverse function \( f^{-1} \) is also a homomorphism, then \( f \) is called an isomorphism. If there is an isomorphism \( V \to V' \), then \( V \) and \( V' \) are said to be isomorphic.

9. Show that all isomorphisms are bijective homomorphisms.
10. Show that all bijective homomorphisms are isomorphisms, proving the following theorem. (Hint: when showing \( f^{-1}(w_1 + w_2) = f^{-1}(w_1) + f^{-1}(w_2) \), for example, let \( u_1 = f^{-1}(w_1) \) and \( u_2 = f^{-1}(w_2) \) which implies \( w_1 = f(u_1) \) and \( w_2 = f(u_2) \). Work from the left hand side to the right hand side using these substitutions.)

Theorem. A function \( f : V \to V' \) between vector spaces (or \( R \)-modules) is an isomorphism if and only if (i) it is a homomorphism, (ii) it is injective (one-to-one), and (iii) it is surjective (onto).

11. Show that if \( u_1, \ldots, u_n \) is a basis for \( V_1 \) and if \( f : V_1 \to V_2 \) is an isomorphism, then \( f(u_1), \ldots, f(u_n) \) is a basis for \( V_2 \). Conclude that if \( V_1 \) is a finite dimensional vector space of dimension \( n \), and if \( V_2 \) is isomorphic to \( V_2 \), then \( V_2 \) is a finite dimensional vector space of the same dimension \( n \). (Optional: does this work for infinite basis? Does this work for modules?)