Problems 1–5: *The trace of a matrix.* Let $A, B \in M_n(R)$ where $R$ is a commutative ring.

**Definition.** Let $A = [a_{ij}]$ be in $M_n(R)$ where $R$ is a commutative ring. Then the *trace* of $A$, written $\text{Tr}A$, is defined to be $\sum_{i=1}^{n} a_{ii}$.

1. Show that $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$. Show that the trace defines an $R$-module homomorphism $M_n(R) \to R$. (Since homomorphisms into $R$ are often called *functionals*, we can call this the *trace functional*.)

2. Show that the trace of $A$ is just $(-1)^{n-1}$ times the $t^{n-1}$ coefficient of the characteristic polynomial of $A$. Conclude that similar matrices have the same trace. (Hint: most $\sigma \in S_n$ give lower power terms.)

3. For matrices in $M_2(R)$ show that you can compute the characteristic polynomial simply by finding the trace and determinant.

4. Suppose that $V$ is a vector space or $R$-module that has a finite basis $v_1, \ldots, v_n$. Define a *trace map* $\text{Tr} : \text{End}(V) \to R$. Show that it is a homomorphism and does not depend on the choice of basis of $V$.

5. Show that if $B$ is in $GL_n(R)$ then $\text{Tr}(AB) = \text{Tr}(BA)$. Hint: look at $B^{-1}(BA)B$. (Optional) Generalize to the case where $B$ is not in $GL_n(R)$. Hint: use the product formula for matrices.

Problems 6–13: *Diagonalization.* Let $V$ be a finite dimensional vector space with scalar field $F$. Let $f : V \to V$ be an endomorphism.

**Definition.** An endomorphism $f \in \text{End}(V)$ is said to be *diagonalizable* if there is an ordered basis $v_1, \ldots, v_n$ such that $\text{Mat}_{(v_i)}(f)$ is a diagonal matrix.

6. Show that $f$ is diagonalizable if and only if $V$ has a basis of eigenvectors.

7. Suppose that $w_1, \ldots, w_k$ are eigenvectors for *distinct* eigenvalues. Show that $w_1, \ldots, w_k$ are linearly independent. Hint: take a non-trivial dependency with the fewest number of non-zero terms. Apply $f$, giving a second dependency. From these two, get a linear dependency with fewer terms.

8. Show that if the characteristic polynomial of $f$ has distinct roots, then $f$ is diagonalizable.

9. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with negative determinant, then $f$ is diagonalizable.

10. Show that if the number of eigenvalues of $V$, counting multiplicity, is $n$ if and only if $f$ is diagonalizable. Show if $f$ is diagonalizable, the associated diagonal matrix is essentially unique: it consists of eigenvalues with the multiplicities giving the number of entries with a given eigenvalue. (Hint 1: choose a basis for each distinct non-trivial eigenvector spaces. Show that the union of these basis is a basis for $V$. Hint 2: to justify the previous hint, show that any non-trivial linear dependency of the vectors can be grouped into a dependency $w_1 + \ldots + w_k = 0$ where $w_1, \ldots, w_k$ are eigenvectors of distinct eigenvalues.)

11. Let $c \in F$ be an eigenvalue of multiplicity $k$. Show that $(t-c)^k$ divides the characteristic polynomial of $f$. In other words, the “algebraic multiplicity” is greater than or equal to the true multiplicity. Conclude (assuming unique factorization of polynomials) that the number of eigenvalues, counting multiplicity, is at most $n$. Hint: form a basis $v_1, \ldots, v_n$ that uses eigenvectors $v_1, \ldots, v_k$. What does $\text{Mat}_{(v_i)}(f)$ look like?

12. Suppose that $\theta$ is not a multiple of $\pi$. Explain why the rotation $R_\theta$ in $\mathbb{R}^2$ fixing $(0,0)$ does not have eigenvectors for $F = \mathbb{R}$. Conclude that $R_\theta$ is not diagonalizable. What if $F = \mathbb{C}$? Show that the trace of $R_\theta$ has absolute value less than $2$, so the characteristic polynomial has distinct complex roots. What are the diagonal entries (eigenvalues) for $\theta = \pi/2$?

13. Let $F = \mathbb{Z}_3 = F_3$. Suppose that, for some basis $(v_i)$ and some $c \in F$,

$$\text{Mat}_{(v_i)}(f) = \begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}.$$

Show that $f$ diagonalizes if $c = 1$, but not if $c = 2$ or $c = 0$. How does $f$ diagonalize if $c = 1$?