Problems 1–2: Transpose. Let $A, B \in M_{m,n}(R)$ where $R$ is a commutative ring. Recall the definition of $A^T$ from LA16:

**Definition (Transpose).** Let $A = [a_{ij}]$ be an $m$ by $n$ matrix. Then the transpose of $A$, written $A^T$, is the $n$ by $m$ matrix with $(i, j)$ entry $a'_{ij}$ given by $a'_{ij} = a_{ji}$.

1. Show that $(A + B)^T = A^T + B^T$ and $(AB)^T = A^T B^T$ whenever the product $AB$ is defined. Thus the transpose operator defines an anti-isomorphism $M_n(R) \to M_n(R)$ which is its own inverse. (An anti-homomorphism $f : R_1 \to R_2$ between rings is a function such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(b)f(a)$ and $f(1) = 1$).

Problems 3–4: Cofactors (Optional). Cofactors occurred in the formula for inverses from LA17 and LA18. Here we discuss a shortcut for computing them (which is the method given in most texts). Let $R$ be a commutative ring and let $V = R^n$. Let $\Lambda : V^n \to R$ be the normalized alternating $n$-linear functional. Then the $(i,j)$th cofactor of a matrix $A \in M_n(R)$ is defined to be $\Lambda(w_1, \ldots, w_{i-1}, e_j, w_{i+1}, \ldots, w_n)$ where $w_1, \ldots, w_n$ are the columns of $A$.

3. Consider the matrix $A'$ obtained by replacing the first column of $A \in M_n(R)$ by $e_1$. Let $A_{11}$ be the matrix obtained by removing the first row and first column of $A$. Show that in the determinant formula for $A'$, you only need to sum over the permutations $\sigma \in S_n$ that fix 1. Show that the set of permutations in $\sigma \in S_n$ that fix 1 forms a subgroup $H \subseteq S_n$ isomorphic to $S_{n-1}$. Compare the $(n-1)!$ terms of det $A_{11}$ with the $(n-1)!$ terms of det $A'$ corresponding to $\sigma \in H$. Show that $\det A' = \det A_{11}$.

4. Let $A = [a_{ij}]$. Use the determinant formula to show that the $(i,j)$th cofactor is equal to $(-1)^{i+j} \det A_{ji}$ where $A_{ji}$ is the $n-1$ by $n-1$ matrix obtained by removing the $j$th row and the $i$th column. Hint: permute the rows and columns of $A$ so that the new matrix $A'$ has first column equal to $e_1$, and so that when you remove the first row and column you are left with $A_{ji}$. Show that $\det A' = \det A_{ji}$.

Problems 5–8: Row Rank and Column Rank. Assume that $R = F$ is a field and that $A \in M_{m,n}(F)$ is an $m$ by $n$ matrix. The dimension of the span of the columns vectors of $A$ is called the column rank of $A$. The dimension of the span of the row vectors is called the row rank of $A$. Our goal is to show that the column rank equals the row rank.

5. Recall the definition of rank from LA10. If $A$ is the matrix of a linear map $f : F^n \to F^m$, show that the rank of $f$ is the column rank of $A$. Show that the column rank is at most the minimum of $m$ and $n$.

6. Show that if $B \in M_n(F)$ is invertible, then $AB$ and $A$ have the same column ranks. Show that if $C \in M_n$ is invertible, then $CA$ and $A$ have the same column ranks. Using transposes, show the same holds for row ranks. Hint: $B$ and $C$ are matrices of isomorphisms. For example, if $A = \text{mat}(f)$ and $C = \text{mat}(\gamma)$, then show that the image $(\gamma \circ f)(F^n)$ is isomorphic to the image $f(F^m)$.

7. Suppose that $A = [a_{ij}]$ is such that (i) if $i \neq j$ then $a_{ij} = 0$, and (ii) for each $i$, either $a_{ii} = 1$ or $a_{ii} = 0$. Show that the column rank of $A$ and the row rank of $A$ are equal. Show that the rank is the number of $i$ such that $a_{ii} = 1$.

8. Using row and column operations, show that there is an invertible matrix $B \in M_n(F)$ and an invertible matrix $C \in M_m(F)$ such that $CAB$ is a matrix $A' \in M_{m,n}(F)$ as in problem 7. Prove the following:

**Theorem.** Let $F$ be a field and $A \in M_{m,n}(F)$ a matrix. Then the row rank of $A$ is equal to the column rank of $A$. This rank is at most the minimum of $m$ and $n$.

**Remark.** The rank of a matrix is defined to be either the row or the column rank: since they are equal, it doesn’t matter which we choose. A matrix in $M_{m,n}(F)$ has maximal rank if its rank is exactly the minimum of $m$ and $n$. So for square matrices, maximal rank means invertible (or non-singular). There is a sense that a random matrix in $M_{m,n}(\mathbb{R})$ has maximal rank.

9. Let $p$ be a prime number. How many matrices are in the ring $M_2(\mathbb{Z}_p)$? How many are invertible? In other words, what is the size of $GL_2(\mathbb{F}_p)$? What is the probability that a random matrix in $M_2(\mathbb{Z}_p)$ is invertible? What happens to this probability as $p$ grows? Optional: generalize to $n > 2$. 

LA 19

Linear Algebra (Spring 2005, Prof. Aitken).

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