LA 17
Linear Algebra (Spring 2005, Prof. Aitken).

Problems 1–2: Non-singular matrices. Let $F$ be a field and let $V = F^n$. Let $\Lambda : V^n \to F$ be the unique normalized alternating $n$-linear functional on $V$. (Normalized means $\Lambda(e_1, \ldots, e_n) = 1$).

1. Let $f : V^n \to F$ be an alternating $n$-linear functional. Show that if $u_n$ is a linear combination of $u_1, \ldots, u_{n-1}$ then $f(u_1, \ldots, u_n) = 0$. Conclude that if the columns of a matrix are linearly dependent, then its determinant is 0. Show that if the rows of a matrix are linearly dependent, then its determinant is 0.

2. Show that if a square matrix $A \in M_n(F)$ has linear independent columns, then $A$ is invertible, and det $A \neq 0$. Hint: see LA10 problems 1 and 2, and LA16 problem 9. Now prove the following:

**Theorem.** Let $A \in M_n(F)$ be a square matrix where $F$ is a field. Then the following are equivalent:

1. The matrix $A$ is invertible. (So $A \in GL_n(F)$). In other words, $A$ is non-singular).
2. det $A \neq 0$.
3. The column vectors of $A$ form a basis for $F^n$.
4. The column vectors of $A$ are linearly independent in $F^n$.
5. The column vectors of $A$ span $F^n$.
6. The row vectors of $A$ form a basis of $F^n$.
7. The row vectors of $A$ are linearly independent in $F^n$.
8. The row vectors of $A$ span $F^n$.

Problems 3–6: Cramer’s Rule. Now we know that det $A \neq 0$ is equivalent to $A$ being invertible, if the scalars are a field $F$. But what if the scalars are a commutative ring? The answer is that $A \in M_n(R)$ is invertible if and only if det $A$ is a unit in $R$. In order to show this, we need Cramer’s Rule, which is an important idea even over fields. Let $V = R^n$ where $R$ is a commutative ring, and let $f : V \to V$ be invertible. Cramer’s rule is a technique to find $f^{-1}(u)$ of a vector $u \in V$ if $f$ is invertible.

3. Explain how the problem of finding $f^{-1}(u)$ is related to solving $n$ linear equations in $n$ unknowns:

$$a_{i1}x_1 + \ldots + a_{in}x_n = c_i \quad i = 1, \ldots, n.$$  

Conclude that Cramer’s rule can be understood as a technique for solving linear equations.

4. Let $w_j = f(e_j)$ and suppose $f$ is invertible. Show that if $f^{-1}(u) = (b_1, \ldots, b_n) = \sum b_ie_i$ then $u = \sum b_iw_i$. Show that $\Lambda(w_1, \ldots, w_{i-1}, u, w_{i+1}, \ldots, w_n) = b_i\Lambda(w_1, \ldots, w_n)$ and prove the following

**Theorem (Cramer’s Rule).** Let $V = R^n$ where $R$ is a commutative ring, and let $\Lambda : V^n \to R$ be the normalized alternating $n$-linear functional for $V$. Assume that $f : V \to V$ is linear and invertible with matrix $A$. Then if $u \in V$, the preimage $f^{-1}(u) = (b_1, \ldots, b_n)$ is given by the formula

$$b_i = (\det A)^{-1}\Lambda(w_1, \ldots, w_{i-1}, u, w_{i+1}, \ldots, w_n)$$

where $w_j$ is the $j$th column of $A$ and $u$ is put in the $i$th slot.

5. Suppose that $f(e_1) = (1, 2, 1)$, $f(e_2) = (3, 1, -1)$, $f(e_3) = (0, 1, 1)$. (Assume $R = \mathbb{Q}$ if you wish). Use Cramer’s rule to find a vector $v$ such that $f(v) = (0, 2, 1)$. (Optional: now find it with row operations).

6. A very important special case is computing $f^{-1}(e_j)$ since it gives the matrix for $f^{-1}$. Prove the following. Use the theorem to find the matrix for $f^{-1}$ where $f(e_1) = (1, 2, 1)$, $f(e_2) = (3, 1, -1)$, $f(e_3) = (0, 1, 1)$. Assume $R = \mathbb{Q}$ if you wish. (Optional: now find it using row reduction).

**Theorem (Inverse formula).** Let $V = R^n$ where $R$ is a commutative ring. Also, assume that $\Lambda : V^n \to R$ is the normalized alternating $n$-linear functional for $V$. Finally, assume that $f : V \to V$ is linear and invertible with matrix $A$ and inverse matrix $B$. Then $f^{-1}(e_j) = (b_{1j}, \ldots, b_{nj})$ is given by the formula

$$b_{ij} = (\det A)^{-1}\Lambda(w_1, \ldots, w_{i-1}, e_j, w_{i+1}, \ldots, w_n)$$

where $w_j$ is the $j$th column of $A$ and $e_j$ is put in the $i$th slot. Also $B = [b_{ij}]$ is the matrix for $f^{-1} : V \to V$ (so $B = A^{-1}$). Thus $A^{-1}$ can be computed with determinants.