Let the image of the unit cube. It turns out, but we will not prove, that for any solid

\[ a = \] 

\[ b = \] 

\[ c = \] 

Definition. Let \( V \) be a vector space or module over \( R \). A \textit{trilinear functional} is a function \( f : V \times V \times V \to R \) which is linear in each coordinate (explain what this means). If \( f(u, v, w) = 0 \) whenever two of \( u, v, w \in V \) are equal, then we say that \( f \) is \textit{alternating}.

Problems 1–6: Basic properties of alternating trilinear functionals. Assume that \( f : V \times V \times V \to R \) is an alternating trilinear functional.

1. Give an example of an alternating trilinear functionals in terms of signed volumes in \( \mathbb{R}^3 \). Hint: generalize parallelogram to three dimensions. (Don’t be too rigorous here).

2. Expand \( f(u, v + w, v + w) \). Conclude that \( f(u, v, v) = -f(u, v, w) \). Generalize: any time two input vectors are switched, then \( f(u, v, w) \) is multiplied by \(-1\).

3. Now assume that \( V = \mathbb{R}^3 \). Let \( u = (a_{11}, a_{21}, a_{31}) \), \( v = (a_{12}, a_{22}, a_{32}) \) and \( w = (a_{13}, a_{23}, a_{33}) \). Show that

\[ f(u, v, w) = \left( a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \right)f(1, 2, 3) \]

Hint: write \( u = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 \) \textit{et cetera}, and expand using linearity.

4. Assume that \( V = \mathbb{R}^3 \) where \( R \) is commutative. Show that the unique normalized (with \( f(1, 2, 3) = 1 \)) alternating trilinear functional is

\[ f\left( (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33}) \right) = \sum_{\sigma \in S_3} \epsilon(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} \]

where \( \sigma \) varies over all 6 permutations of 1, 2, 3, and where \( \epsilon(\sigma) \) is +1 or −1 depending on \( \sigma \).

Definition. Let \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear map. Define the \textit{determinant} of \( L \) to be the oriented volume of the image of the unit cube. It turns out, but we will not prove, that for any solid \( C \) in \( \mathbb{R}^3 \), that the volume of \( L(C) \) is the absolute value of the determinant of \( L \) times the original volume of \( C \).

5. Show that if the matrix of \( L \) is \([c_{ij}]\) then the determinant of \( L \) is

\[ \sum_{\sigma \in S_3} \epsilon(\sigma)c_{1\sigma(1)}c_{2\sigma(2)}c_{3\sigma(3)} \]

6. Use this formula to calculate volumes of some solids in \( \mathbb{R}^3 \).

Definition. Let \( V \) be a vector space or module over \( R \). An \textit{n-linear functional} is a function \( f : V^n \to R \) which is linear in each coordinate: if \( u_i = v_i + w_i \) then

\[ f(u_1, \ldots, u_i, \ldots, u_n) = f(u_1, \ldots, v_i, \ldots, u_n) + f(u_1, \ldots, w_i, \ldots, u_n) \]

and, if \( u_i = cw_i \) with \( c \in R \),

\[ f(u_1, \ldots, u_i, \ldots, u_n) = cf(u_1, \ldots, w_i, \ldots, u_n). \]

If \( f(u_1, \ldots, u_n) = 0 \) whenever \( u_i = u_j \) for \( i \neq j \), then we say that \( f \) is \textit{alternating}.

Problems 7–9: Basic properties of alternating n-linear functionals. Assume that \( f : V^n \to R \) is an alternating \( n \)-linear functional. Review permutations from abstract algebra, if necessary.

7. Show that if \( \sigma \) is a transposition (2 cycle), then \( f(u_1, \ldots, u_n) = -f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \).

8. Show that if \( \sigma \) is an odd permutation, then \( f(u_1, \ldots, u_n) = -f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \). Show that if \( \sigma \) is an even permutation, then \( f(u_1, \ldots, u_n) = f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \). (The group of even permutations is called the \textit{alternating group}: alternating functionals are invariant under the alternating group).

9. Now consider the case where \( V = \mathbb{R}^n \). Suppose that \( w_1, \ldots, w_n \) are vectors (often \( e_1, \ldots, e_n \)). Suppose \( u_j = a_{ij}w_1 + \ldots + a_{nj}w_n \). Show that when you expand \( f(u_1, \ldots, u_n) \) using linearity, you get \( n^n \) terms (some will turn out to be zero). To choose a term of the expansion, pick a term \( a_{\gamma(j)} w_{\gamma(j)} \) from each \( u_j = \sum a_{ij}w_i \), where \( \gamma(j) \in \{1, \ldots, n\} \). Note there are \( n^n \) ways of choosing \( \gamma(1), \ldots, \gamma(n) \) and

\[ f(u_1, \ldots, u_n) = \sum_{\gamma} f(a_{\gamma(1)}w_{\gamma(1)}, \ldots, a_{\gamma(n)}w_{\gamma(n)}) \].