Problems 1-7: Inverses of matrices. In problems 1-4 assume \( R \) is a field or a commutative ring. In problems 5-7 we will play it safe and assume that \( R = F \) is a field. Use the ring isomorphism \( \Phi : M_n(R) \to \text{End}(R^n) \).

1. Let \( A \in M_n(R) \). Then \( A \) invertible means it is has a multiplicative inverse in the ring \( M_n(R) \). In other words, there exists a matrix \( B \in M_n(R) \) such that \( AB = BA = I \) where \( I \) is the identity matrix. From ring theory we know that the inverse of \( A \), if it exists, is unique and \( (A^{-1})^{-1} = A \). Use the isomorphism from \( \Phi : M_n(R) \to \text{End}(R^n) \) to observe that if \( A = \text{Mat}(f) \) then \( B \) is the inverse of \( A \) if and only if \( B = \text{Mat}(f^{-1}) \).

So inverting matrices is equivalent to finding inverses of linear maps.

2. Show that if a matrix in \( M_n(R) \) has a zero row, or a zero column, then it is not invertible. Hint: interpret the problem in terms of linear maps that can be shown not to be surjective or not to be injective.

3. Show that if \( f : A \to B \) and \( g : B \to C \) are functions with \( g \circ f \) equal to the identity map, then \( g \) is surjective and \( f \) is injective. (This is just set theory: no linear algebra is required).

4. Show that if \( A, B \in M_n(R) \) are matrices with \( AB = I \) with \( I \) the identity matrices, then \( A \) is the matrix of a surjective linear map and \( B \) is the matrix of an injective linear map.

5. (Continued) Show that \( A \) and \( B \) are matrices of isomorphisms (hint: use \( \text{LA10, problem 11} \)). Conclude that \( A \) and \( B \) are invertible.

6. Show that if \( A, B \in M_n(R) \) are matrices with \( AB = I \) with \( I \) the identity matrices then \( A \) and \( B \) are inverses of each other (in other words, that \( BA = I \)). Hint: you know that \( A^{-1} \) exists. Show \( B = A^{-1} \).

**Proposition.** If \( A, B \in M_n(F) \) are such that \( AB = I \) then \( BA = I \). (Here \( F \) is a field, or integral domain).

7. Generalize. Show that if \( A, B \in M_n(F) \) are matrices with \( AB \) invertible, then both \( A \) and \( B \) are invertible.

Problems 8-14: Row and column operations. Assume \( R \) is a field or a commutative ring.

8. Suppose that \( 1 \leq i < j \leq m \). Show that there is a matrix \( X \in M_{m,n}(R) \) such that, for all \( n \) and for all \( A \in M_{m,n}(R) \), the matrix \( XA \) is equal to the matrix \( A \) except that the \( i \)th and \( j \)th rows are switched. Show that \( X \) is its own inverse. Show that if \( A \in M_{m}(R) \) then \( A \) is invertible if and only if \( XA \) is invertible. (General fact: for any associative operation with identity element, if \( x \) is invertible, then the product \( xa \) is invertible if and only if \( a \) is.)

9. Give a similar construction for switching columns.

10. Let \( 1 \leq i \leq m \) and let \( c \) be a unit in \( R \) (so if \( R \) is a field, just assume \( c \neq 0 \)). Find a matrix \( X \in M_{m}(R) \) such that, for all \( n \) and for all \( A \in M_{m,n}(R) \), the matrix \( XA \) is equal to the matrix \( A \) except that every entry the \( i \)th row has been multiplied by \( c \). Describe an inverse for \( X \). Conclude that if \( A \in M_{m}(R) \) then \( A \) is invertible if and only if \( XA \) is invertible. Give a similar construction for changing a column.

11. Let \( i \) and \( j \) be distinct row numbers, and let \( c \) be an element of \( R \). Find a a matrix \( X \in M_{m}(R) \) such that, for all \( n \) and for all \( A \in M_{m,n}(R) \), the matrix \( XA \) is equal to the matrix \( A \) except that the \( i \)th row (considered as a vector in \( R^n \)) has been replaced by the \( i \)th row plus \( c \) times the \( j \)th row. Show that \( X \) has an inverse. Conclude that if \( A \in M_{m}(R) \) then \( A \) is invertible if and only if \( XA \) is invertible. Give a similar construction for changing a column in this way.

12. The above three operations are called row operations and column operations. Consider the following procedure. Start with \( C := A \) and \( D := I \). Then in the row operation step choose a row operation, and perform this operation to both \( C \) and \( D \): in other words replace \( C \) by \( XC \) and \( D \) by \( XD \) for \( X \) as above. Repeat the row operation step until \( C = I \). Show that if this procedure ends then \( D = A^{-1} \). Hint: show that \( D \) is invertible at each stage, and that at each stage \( D^{-1}C = A \).

13. Find inverses of matrices of your choosing in \( M_m(Q) \), \( M_m(F_3) \) and \( M_m(Z) \), using this technique.

14. Describe a procedure which, given \( A \in M_n(F) \), will either find an inverse for \( A \) or show you that \( A \) is not invertible. Assume \( R = F \) is a field for this. Hint: if there is a dependency among columns, then \( C \) is not invertible. Why?