Study Guide for Exam 3

Math 330: History of Mathematics


1 Introduction

What follows is a list of topics that might be on the exam. Of course, the test will only contain a selection of these: there is simply too much to put on one exam. (Thus, some of the topics will appear on the final instead, and some topics will appear on this exam and the final). Read this study guide carefully since there are a few things here that we did not have time to get to in class, but still might be on the test.

History is full of names of people, place names, and dates. Don’t worry: the only personal names, place names, or dates you need to memorize are given in this study guide. (Footnotes and parenthetical remarks in this study guide are given for information and clarification. You do not need to know them for the exam.)

2 Mathematicians

1. Ahmes the scribe. Lived after the Middle Kingdom of Egypt and before the New Kingdom (about 1650 BC). His is the earliest known name in the history of mathematics. Author of the famous Rhind-Ahmes Papyrus, the most extensive mathematical papyrus yet found. (Since Egyptian writing only recorded consonants, not vowels, there is some ambiguity concerning his name. Perhaps his name was Ahmose).

2. Al-Khwarizmi (c. 780 - c. 850). One of the earliest and most important Arabic mathematicians. Worked at the famous House of Wisdom which was located in Baghdad. At the time, Baghdad was a new city, and was the capital of the Islamic world.

The word “algebra” comes from the title of his most famous work (Hisab al-jabr w’al-muqabala concerning calculation using the operations of al-jabr and al-muqabala. The term al-jabr means “completion” or “restoration”. It refers to adding to both sides of an equation to compensate for a subtraction. For example, going from \(x^2 - 5 = x\) to \(x^2 = x + 5\). The term al-muqabala means “balancing”. It refers to the operation of canceling from both sides of the equation. For example, going from \(x^2 + 3x = 5x + 1\) to \(x^2 = 2x + 1\). With these two operations every quadratic equation can be reduced to one of al-Khwarizmi’s standard forms.) This book discusses how to solve quadratic equations, but does not use symbolism: everything is written out in words. It justifies the procedures using geometric arguments: when al-Khwarizmi completes the square, he really completes the square. Of course, quadratic problems had been solved long before al-Khwarizmi, but he lays things out in a very organized, practical, and accessible way. Because of this his book was very influential. This book was
very influential in Europe as well.\(^1\) (This book also discusses linear equations, areas and volumes, and algebraic problems arising from dividing inheritances.)

Another important book, which we only know about through its Latin translation, is *Algoritmi de numero Indorum*. The word *Algoritmi* is a Latinized version of Al-Khwarizmi’s name. This translated book was influential in Europe in spreading Hindu-Arabic numbers. It discusses how to calculate with these numbers, which were new at the time. It is from this book that we derive our word *algorithm*.

(Al-Khwarizmi also wrote on astronomy and trigonometry, based on Indian sources, geography, the sundial, the astrolabe, the Jewish calendar, and history. His geography improves on Ptolomy’s).

3. **Gerbert and the Medieval Translators.** Gerbert (who lived about the year 1000) was one of the earliest Medieval era Europeans to show an interest in mathematics. He studied in Spain, and knew about the Arabic contributions to mathematics. His math books were elementary by our standards, but they stimulated interest in mathematics. Gerbert later became the Pope. Because of his influence, there were several famous scholars who went to Spain during the 12th century to translate mathematical works from Arabic to Latin. Later translators translated Greek works into Latin. (Some of the Arabic writings were themselves translations of Greek works, so the Latin translations were two steps removed from the original source). Latin was the universal language of scholars of Western Europe, just as Greek was the universal scholarly language of Eastern Europe, and Arabic was the universal scholarly language of the Islamic world. The scholarly language of India was Sanskrit.

4. **Omar Khayyam** (1048-1122). Famous Persian poet, mathematician, scientist, and philosopher. (His Arabic name was al-Khayyami). Wrote mathematics in Arabic but poetry in Persian. He also wrote a book on *al-jabr* the first to tackle general cubic equations. His solutions were geometric: involving the intersection of curves such as conics and circles. (He wrote this book in Samarkand, a famous ancient city now in Uzbekistan).

He wrote a critique of Euclid (discussing the fifth postulate and the theory of proportion). (He also reformed the calendar based on his accurate estimates for the length of a solar year. His idea was to have 8 leap years every 33 years. He is famous worldwide outside of mathematics for writing the poem called the *Rubaiyat*).

5. **Bhaskara** (1114-1185) Indian mathematician and astronomer. Wrote the *Lilavati* which was dedicated to his daughter Lilavati to consol her. Why did she need consoling? Thought that \(a/0 = \infty\). He represents the high point of Indian mathematics

6. **Fibonacci**. (1170-1250) His real name was **Leonardo of Pisa**. An Italian mathematician, the most famous and important medieval European mathematician. From a merchant family, he travelled widely in the European and Arabic worlds, and learned some Arabic mathematics. He advocated for the Hindu-Arabic numerals instead of the then current Roman numerals. Despite the efforts of Fibonacci and others, Roman numerals remained popular for several more centuries. He is most famous for his Fibonacci sequence motivated by a problem involving the population of rabbits. What are the Fibonacci numbers? How are they formed?

\(^1\)It was translated by Robert of Chester into Latin in 1145.
His performance at a mathematics contest made him famous. (For example, he showed that the cubic \( x^3 + 2x^2 + 10x = 20 \) cannot be solved with square roots, but he gave a very accurate approximate solution 1.3688081075, good to 9 places. He also found a rational square such that if you add or subtract five, you still have a rational square: \((41/12)^2\). He realized, as did others, that if \( p \) and \( q \) are the sum of two squares, then so is \( pq \).)

7. **Yang Hui** (c. 1238 - c 1298) Chinese mathematician who wrote several outstanding mathematical texts including a commentary on the Nine Chapters. This commentary discusses the binomial theorem and Pascal’s triangle. This is an example of the use Pascal’s triangle long before Pascal. He also wanted to get people excited about mathematics by using things like magic squares and magic circles. Some of his magic squares were very extensive.

8. **Oresme** (1323 - 1382). A French mathematician, scientist, and philosopher. Later a bishop. Friend to the King of France. He did three important things, all of which were ahead of his time: (i) developed fractional exponents such as \( 8^{2/3} \) (not with this notation), (ii) developed the idea of a graph of a function displaying the independent variable (the input) as the first coordinate and the dependent variable (the output) as the second coordinate (which may have influenced Descartes), and (iii) argued that the harmonic series

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots
\]

diverges. What was his proof of this divergence?

9. **Regiomontanus**. German mathematician of the early Renaissance. (His real name was Johann Müller). The best mathematician and astronomer of his generation. The first European to treat trigonometry as its own subject, and not just as a part of astronomy. (The Arab mathematician Nasir Eddin did this earlier). He learned ancient Greek in order to translate the Greek mathematical and astronomical classic. He even set up a printing press.

10. **Albrecht Dürer and the Renaissance artists**. Not only mathematicians, but artists became interested in mathematics during the Renaissance. For example, the artist Leonardo da Vinci became interested in the golden ratio, and the artist Dürer, from Germany, used a famous magic square in his art. These two artists, and many others, became interested in geometry, especially the theory of perspective. Two important perspective facts: (i) parallel lines appear to intersect in a point on the horizon, (ii) circles appear to be ellipses or other conic sections. In fact, parabolas can appear, in perspective, to be ellipses!

11. **Pacioli**. Renaissance Italian mathematician. A roommate of the artist, Leonardo da Vinci. He taught Leonardo mathematics. Leonardo drew illustrations for Pacioli’s book *De divina proportione*. This book is responsible for modern interest in idea of the golden ratio. Wrote the first *published* algebra book called the *Summa* (1494). It has algebraic symbolism that became popular for a while, but different than our modern notation. It also had arithmetic, some elementary geometry, and bookkeeping!

12. **Robert Recorde and the “Cossic Artists”**. The term *coss* is German for “unknown” (the Italian term was *cosa* meaning “thing”). For much of the Renaissance, algebra was called the *cossic art*. Most “cossic artists” were German. The cossic artists were responsible for much of our modern symbolism including + and − and \( \sqrt{\quad} \). Robert Recorde was an English
“cossic artist” who was responsible for the symbol =. He chose this since “no two things can be more equal” than parallel lines of the same length. Recorde was also a physician to royalty (including King Edward VI, and Queen Mary).

13. **Thomas Harriot** (1560-1621). English mathematician of the late Renaissance. Science advisor of Sir Walter Raleigh (where his duties included ship design, developing and teaching navigational techniques, surveying, keeping the financial accounts, and solving various mathematical problems put to him.) He was one of the first Englishmen to travel to North America: he was on Raleigh’s expedition to Virginia (1585-86, perhaps also to Roanok Island 1584. He died of cancer of the nose caused by his habit of inhaling tobacco smoke, a practice he learned in Virginia.)

In algebra he introduced the symbols > and <. Harriot advocated moving all terms of an algebraic equation to one side of the equation and setting this equal to zero (he was very comfortable with negative numbers). He made several other advances in algebra as well.

He came up with an interesting formula for the area of spherical triangle. Let $H$ be the area of a hemisphere, and $T$ the area of a triangle on a sphere. Then

$$\frac{T}{H} = \frac{E}{360}$$

where $E$ is the excess angle sum: the sum of the three angles minus 180 degrees.

(He made observations with telescopes about the same time as Galileo: mountains on the moon, moons of Jupiter, and sunspots — he even found the period of rotation of the sun based on the sunspots. His interests included algebra, astronomy, optics, chemistry, trajectories of projectiles, the rainbow.)

14. **Scipione de Ferro** Renaissance Italian mathematician. First mathematician to come up with an algebraic method of solving cubic equations. Perhaps only solved only one type of cubic equation (namely those of the form $x^3 + mx = n$). He never published his work but kept it secret. He revealed the method to his student (Antonio Fior).

15. **Tartaglia** Renaissance Italian mathematician. Came up with an an algebraic method of solving cubic equations. His methods worked for more types of cubics than the solution of Scipione de Ferro (including cubics of the type $x^3 + px^2 = n$). There was a famous contest between Tartaglia and the student (Fior) on solving cubics, and Tartaglia won decisively. Later Tartaglia became incensed with Cardano when Cardano published his method.

16. **Cardano** (1501-1576). Famous Renaissance Italian physician, mathematician, notorious gambler, and astrologer (at one time he was an astrologer to the Pope). He convinced Tartaglia into telling him his method for cubic equations (1539). Cardano swore he would not tell anyone. Yet in his influential algebra book *Ars magna* “the great art” (1545) he explained how to solve cubics. (Cardano’s student, Ferrari, claimed that Cardano also learned it from another source, and then extended it in original ways so he could legitimately write about it.) The *Ars magna* also has a solution to the fourth degree polynomial discovered by his student Ferrari. (It was only after 1800 that mathematicians such as Abel were able to show that general polynomials of degree five or more cannot be solved with these types of algebraic methods. He also was a forerunner to probability, due partly to his gambling.)
17. **Ferrari** Renaissance Italian mathematician. Student of Cardano. First person to solve the fourth degree equation.

18. **Bombelli** (1526-1572). Renaissance Italian mathematician. Wrote a famous algebra book that explains the role of imaginary and complex numbers in solving cubic and other types of equations. (The book is called *Algebra* and it was published in 1572.)

19. **Viète.** *Introduction to the Analytic Arts.* French Renaissance. He started out as a lawyer but spent most of his career in government under the French king. One of his claims to fame was deciphering the Spanish code for the French King. He wrote a landmark algebra book; Viète called algebra the *analytic arts* since it sounded more prestigious and more Greek-like than the “cossic art”. For a while, the word “analysis” meant algebra, but later shifted meaning to mean infinite algebra, such as series, and advanced Calculus. Viète introduced the idea of having more than one variable, some standing for unknowns and others for parameters. He used distinct letters for distinct unknowns. (He came up with the first exact formula for $\pi$; it involves an infinite product.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \ldots$$

Of course, to get a formula for $\pi$ itself you need to invert both sides, and multiply by 2.)

20. **Copernicus** From Poland. The most famous Renaissance astronomer. He contributed to trigonometry as well. He revolutionized astronomy with the heliocentric theory. (It had not been proposed seriously since Aristarchus). This theory, now the basis for our idea of the Solar System, was revolutionary and controversial at the time.

21. **Galileo.** Italy in the late Renaissance. First to use the telescope in astronomy. Discovered the moons of Jupiter, and sunspots on the sun. Discovered the constant acceleration principle (for objects reasonably close to the surface of the earth, and disregarding air resistance); he used objects sliding on incline planes to slow down the acceleration in order to study it more carefully. He viewed mathematics as an essential tool for understanding the natural world. Discovered that the period of the pendulum makes a excellent timekeeper, and that trajectories of projectiles are parabolas. Persecuted by the inquisition for his advocacy of Copernicus’s controversial heliocentric theory.

22. **Kepler.** Late Renaissance. He was a big fan of Copernicus’ theory. He tried to explain the paths of the planets in several flawed ways (including comparing inscribed and circumscribed circles). Finally he hit on his three laws. What are the three laws of planetary motion that he discovered? Are they based on observational data, or only speculation? Answer: data. Did he explain these laws in terms of inverse square laws (or did that come later)? Answer: that came later with Newton.

23. **Descartes.** (René Descartes. 1596–1650). French philosopher and mathematician. One of the most famous philosophers of all times. He wrote the *La géométrie* as one of the appendices to his most important philosophical work (*Discours de la méthode*). *La géométrie* introduced coordinate geometry. His notation was almost modern: used $x, y, z$ for variables, and $a, b, c$ for constants. He used $x^n$ for the $n$th power. He did not really introduce the so called “Cartesian”
coordinate system, but did take the first important steps. In fact, he did not consider points with negative coordinates. He thought of $x$ and $y$ as lengths, and the point determined by $x$ and $y$ was drawn by drawing $x$ horizontal from then origin, then a segment of length $y$ was drawn at an angle (usually 90 degrees) from the endpoint of the $x$ length. When the angle is 90, this gives the same point that we get in modern coordinate geometry, but the thought process is somewhat different. He interpreted multiplication in a more modern way where a length $A$ times another length $B$ is a third length. Before his time, most mathematicians regarded $AB$ as a rectangle, $A^3$ as a cube, and $A^n$ for $n > 3$ was considered nonsensical. But from Descartes’ time, they are all lengths, so one should not be afraid to take large powers.

While in school he was allowed to stay in bed thinking until 11 in the morning. He kept this habit for much of his life (even when he was in the military). He spent much of his productive years in Holland. He moved to Sweden to work for the Queen (Queen Christina). She wanted him to get up at 5AM to give her lessons. That and the cold climate was hard on him: he died in a few months.

24. **Fermat.** (Pierre de Fermat. 1601–1665) French Lawyer working in the city parliament. Did he get paid for his mathematical research? Answer: no. Did he correspond with other mathematicians? Answer: yes, that is the main way in which his discoveries spread.

Fermat together with Descartes is considered the inventor of coordinate or analytic geometry. Fermat’s approach was more understandable than Descartes: he started with linear equations and lines, then built up to quadratic equations and conics. (Descartes basically skipped lines and their linear equations).

Fermat was a key predecessor to calculus. He used infinitesimal methods to find tangents to curves and areas under curves. However, he did not realize that tangents problem and the area problem (quadrature problem) were connected: the Fundamental Theorem of Calculus had to wait until later.

Fermat read Diophantus, and so became interested in number theory. In some ways, Fermat can be regarded as the inventor of number theory. What primes are the sum of two squares? Answer: a prime $p > 2$ is the sum of two squares if and only if $p$ is one more than a multiple of four.

25. **Cavalieri.** (Bonaventura Cavalieri, 1598-1647). An Italian Monk who became a disciple of Galileo. A key predecessor for integral calculus: he developed his method of indivisibles for finding areas and volumes which is related to the method of exhaustion of Archimedes and Eudoxus. His method was not rigorous, but it produced correct answers fairly easily. So some people criticized his method, while others including Galileo were very impressed. Cavalieri thought of a volume as an infinite stack of areas, and an area as a stack of lines. A simple version of Cavalieri’s Principle: if two solids have cross sections at each height with equal area, then they have the same volume. (A version holds for areas with equal length cross sections. One can change ‘equal’ to ‘in a fixed proportion’ to get a more general statement.)

26. **Isaac Newton** 17th century England (specifically 1643-1727, or by the old calendar: born on Christmas 1642). One of the two founders of calculus. His theory was built around “fluons” and “fluxions”. It was not considered as notationally satisfying as Leibniz’s system. He knew the relationship between his version of the theory of derivatives and area under a curve, the
Fundamental Theorem of Calculus. Newton also discovered the binomial series, and many other things. Although he is remembered today primarily for his physics, his mathematical accomplishments are enough to make him one of the greatest mathematicians of all times.

His theory of gravity is explained in his book, the Principia (1687) which is probably the most important scientific book ever published. In it he formulated his inverse square law. He showed that all three of Kepler’s laws follow from the inverse square law. The thing people found strange about gravity is the idea of “force at a distance”. He also formulated laws of motion, and made contributions to the theory of light and optics, especially the idea that white light is a mix of colors (and that different colors are refracted at a different angle).

In 1665, while he was a student at Trinity College in Cambridge University, the University closed due to the plague. It did not open until 1667. Newton went home and had a few years of time to reflect on his interests. In that period he developed the seeds for his theory of gravity and for calculus. (He wrote up his ideas about calculus in 1671, but it wasn’t until 1736 that they were published. They were spread long before 1736, however, through letters, lectures, word of mouth, etc.)

27. Leibniz. 17th century Germany. (More specifically, Gottfried Leibniz, 1646-1716). Like Descartes, he is famous today both as a philosopher and a mathematician. One of the two founders of calculus (the other is Newton). He invented the integral sign \( \int \) as a kind of letter \( S \) for a kind of infinite sum of infinitesimals (1675), and \( dx \) and \( dy \) for infinitesimal differences. Infinitesimals are infinitely small quantities. According to him, the ratio of two infinitesimals is often not infinitely small, but is often an ordinary quantity. What does the ratio \( dy/dx \) mean in that case? Leibniz did not think in terms of limits. So his conception of derivative is different than ours. He gave the basic rules of derivatives (1675-76) and integrals, and knew the inverse relationship between them: the Fundamental Theorem of Calculus. Finding good notation was very important to Leibniz.

In philosophy, Leibniz is responsible for the idea that we are living in the best possible (but far from perfect) universe.

He was more than just a philosopher and a mathematician. His interests and contributions were very broad (including law, logic, physics, Latin poetry, religion, theology, diplomacy, calculating machines, geology, languages including Latin, Greek, and Sanskrit.) He built a calculating machine: the first that could execute all four operations (1694). This machine used binary. (Pascal’s early machine could only add and subtract, and didn’t use binary).

There was a bitter dispute between Newton and Leibniz on who invented the main ideas of calculus. (Leibniz first published his ideas in 1684. Newton’s Principia was published 1687, but Newton had actually written on calculus in 1671 before Leibniz started working on calculus in 1673. Newton’s 1671 paper was not published until 1736, but Newton’s ideas were spread through letters, lectures, etc.)

Leibniz independently discovered the following formula (1674?):

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots
\]

which when multiplied by 4, gives a formula for \( \pi \). (Other work in mathematics by Leibniz includes sums of infinite series, the multinomial theorem, theory of determinates, osculating
circles and envelopes. His idea of an algebra or calculus of rational thought was an important predecessor of modern mathematical logic.)

3 Historical People and Events

The history of mathematics, and the history of science in general, can be divided into four periods: Ancient, Medieval, Renaissance, and Modern. The Ancient period lasts until the fall of the Roman Empire. In the Roman Empire, most mathematics was done by Greek writers, often in Alexandria. The Medieval Period includes Chinese, Indian, Arabic, Mayan, and European accomplishments. At the beginning of this period, the (western) Roman Empire had fallen, and western Europe is often considered to be in the “dark ages”. The eastern Greek speaking half of the Roman Empire remained and evolved into the Byzantine Empire. Their mathematics was not of the highest rank, but at least they kept the tradition alive. The term Renaissance is used mainly to denote the flowering of culture that occurred in Italy and other parts of western and central Europe. Around 1600 the Renaissance ends and the modern period begins.

You will need to know the following, including the dates.

1. **Beginning of urban civilization and writing** in Egypt and Mesopotamia. Before 3000 BC.

2. **Ancient Mathematics** About 1800 BC. The time of the Middle kingdom of Egypt. This is the period of Egyptian history responsible for most known Egyptian mathematics. Ahmes wrote his papyrus a bit later (about 1650 BC, after the Middle Egyptian period), but based on a previous manuscript from the Middle Egyptian period. This is also the time of the height of Babylonian mathematics. Hammurabi was the most famous Babylonian ruler from this period. He is famous for Developing a legal code.

3. **476: The fall of the (western) Roman Empire**. The eastern Roman Empire lasted for almost 1000 years after this, and evolved into the Byzantine Empire. Rome was conquered by Germanic tribes.²

4. **The development of the Hindu-Arabic number system** and the positional decimal system. From about 600-800. Until Al-Khwarizmi, our information is very sketchy.

5. **The growth of Islamic culture**. During the so called dark ages in Europe, Islamic culture prospered. The center of culture was Baghdad (although the religious center was Mecca). The House of Wisdom was in Baghdad. Baghdad was founded in 766. The house of Wisdom was set up in Baghdad about 50 years later.

6. **Renaissance** in Europe. Know the date 1450: the approximate date of the fall of Constantinople (eastern Roman empire) and the beginning of printing in Europe. The height of the Renaissance (in Italy and Central Europe. It would wait to come to England until the late 1500’s).

²England was also conquered by Germanic tribes, the Angles and the Saxons. The English language is a Germanic language, but it has a huge amount of French and Latin influences as well. In fact, English draws from many languages for its vocabulary. We have seen the importance of Greek words to our English vocabulary.
7. **1600.** Approximate end of the Renaissance, and the start of the modern period. Some people date the start of modern period with Descartes.

8. **1665-66.** The plague years when Newton discovered calculus and the laws of gravity.

### 4 Summary of Dates

Know the following dates and their significance.

- 3000 BC
- 1800 BC
- 600–800 (Hindu-Arabic numerals)
- 766 (Baghdad, House of Wisdom)
- 1450 (approximate dates of printing press, the Renaissance, fall of Constantinople).
- 1600. (approximate end of the Renaissance, and the start of the modern period.)
- 1665-66. The plague years.

### 5 Other Historical Questions

Be able to answer the following:

1. What are the three major periods of ancient Egyptian history?
2. In which period were the pyramids made?
3. Which period was responsible for the height of Egyptian mathematics? [Middle Kingdom]
4. Name the two major Egyptian scrolls.
5. What were these scrolls made of? [Papyrus]
6. Which of these major Egyptian scrolls has the unit fraction table? [Rhind]
7. Which Egyptian scroll has the truncated pyramid problem? [Moscow]
8. What inscription led to the decipherment of hieroglyphics? [Rosetta Stone]
9. What is the major river of Egypt? What are the major rivers through Mesopotamia?
10. What does *mesopotamia* mean? What do *meso* and *potamos* mean?
11. Know the difference between *hieroglyphics* and *hieratic*.
12. Which of these two scripts were used in the Ahmes and Moscow Papyri? [hieratic]
13. Which language were the original cuneiform written in? [Sumerian]
14. Which were cuneiforms written on? [Clay tablets]

15. Which language were the mathematical cuneiform and the other writers of the Old Babylonia period written in? [Akkadian]

16. What is the House of Wisdom? Where was it located?

6 Numbers and Letters

1. Know how to write sexigesimal (base 60) numbers in cuneiform, as well as our system. Example: translate base 60 12, 34; 14 into cuneiform. Translate 12, 34; 14 into modern decimal. Translate 15.25 into base 60.

2. Know how to write numbers up to 999 in hieroglyphics.

3. Know how to write numbers using the Mayan system. Know that the positional values are 1, 20, 20 · 18, 20² · 18, 20³ · 18, etc. Remember to write these numerals vertically. Know that dates were recorded based on counting days from day zero (12 August 3113 BC).

4. Know the origins of the words algebra, algorithm, cipher, and sine.

5. Know the scholarly languages: The scholarly language of Babylon was Akkadian. The scholarly language of Egypt was Egyptian. The scholarly language of China was Chinese. The scholarly language of the Greek world, Alexandria, and much of the Roman Empire was Greek. (The official language of the Roman Empire, at least in Italy, was Latin). The scholarly language of India was Sanskrit. The scholarly language of the Islamic world was Arabic. The scholarly language of Medieval and Renaissance Europe was Latin. However, after the printing press, many of the Cossic artists published in their native languages.

7 Mathematical Topics

1. Egyptian Arithmetic. Egyptian multiplication and division using doubling. For example, multiply 104 and 5 using the method of doubling. Divide 261 by 9 with a similar method.

2. Egyptian Fractions. Understand Egyptian (or unit) fractions. Here are some problems that you should be able to do: (1) convert 7/9 or 14/5 into distinct unit fractions \(\pi\). (2) multiply the fraction \(\pi\) by 14. (3) Add \(\psi\) \(\frac{3}{4}\) and \(\frac{3}{4}\) \(\frac{7}{2}\). (4) Divide 104 by 5 and write the answer in terms of distinct unit fractions. For these problems, you may use the unit fraction table found in the Ahmes-Rhind Papyrus.

3. Egyptian Geometry. Know the Egyptian method for finding the volume of the truncated pyramid (also called a frustum). Which papyrus was it in?

4. Egyptian Geometry. How did Ahmes calculate the area of a circle? Answer: divide the diameter by 9, multiply by 8, and square. What value of \(\pi\) results from this approximation? Justify your answer (using \(\pi r^2\)).
5. **Babylonian Numerals.** Be able to use base 60. Give examples of how the Babylonians replaced division by multiplication (using a table of reciprocals for hard situations). For example, to divide 42 by 5, just multiply 42 by 0;12. Do additions and subtractions in base 60. You do not need to do an explicit multiplication in base 60, but you do need to know that (i) multiplication tables were used, and (ii) to divide you just multiply by the reciprocal.

6. **Sexigesimal Numbers.** Convert from base 60 to base 10. For example, what is 12,30,45 in base 10? What is 1;23,30 in base 10? Convert from base 10 to base 60. For example, what is 12/80 in base 60? What is 13451 in base 60?

7. **Babylonian Numerals.** Is the Babylonian method of writing numbers ambiguous? Can different numbers be written the same way?

8. **Babylonian Algebra.** Suppose that you are told \( s = x + y \) and \( a = xy \) but you are not told \( x \) and \( y \). Show how to find \( x \) and \( y \) using the Babylonian method.

On the other hand, we have no evidence that the Egyptians could solve quadratic equations involving three terms.

If you know the area and perimeter of a rectangle, then the above can be used to find the sides. In fact, the language of the Babylonians suggest that they thought of \( a = xy \) as an area, and \( x, y \) as lengths.

9. **Babylonian Algebra.** Suppose that you are told \( d = x - y \) and \( a = xy \). Show how to find \( x \) and \( y \) using the Babylonian method. (I am using \( x \) and \( y \) for convenience: Babylonian algebra was not symbolic.)

10. **Babylonian Algebra.** Give a geometric justification for the formula

\[
m^2 = a + \Delta^2
\]

of Babylonian algebra where \( m \) is the arithmetic mean of \( x \) and \( y \), where \( a \) is the product of \( x \) and \( y \), and where \( \Delta \) is the difference between the variables and the mean (\( \Delta = x - m = m - y \) where \( x > y \)). Assume for convenience that \( x > y \) (although the formula is trivially true if \( x = y \)).

Hint: Do so as follows. Draw three shapes: (1) a rectangle with dimensions \( m \) by \( y \), (2) a rectangle with dimensions \( \Delta \) by \( y \), (3) a square with dimensions \( \Delta \) by \( \Delta \). Now show how these can be packed together as a square with area \( m^2 \). Finally show how these can be packed together as a rectangle with area \( a = xy \) with a square of \( \Delta^2 \).

11. **Zero.** The Babylonians used a space to indicate a zero in a given sexagesimal position, but it was sometimes hard to see where a space was supposed to be. Later Babylonians used a dot to indicate such a space. Greek astronomers including Ptolomy used a symbol for zero. Indian mathematicians treated zero as a number by itself, and not just as a placeholder. Brahmagupta and others gave rules for using zero. Of course, zero became an essential ingredient of the Hindu-Arabic number system (base ten positional). However, most civilizations outside of India treated zero as just a placeholder until after the Renaissance. Harriot (and Descartes) changed all that by advocating solving polynomial equations by moving all the terms to one side and setting them equal to zero. This way of finding roots shows how useful
zero is in algebra. The Mayan’s independently invented zero as a placeholder in their base 20 system

12. **Negative numbers.** Chinese and Indian mathematicians were comfortable, more or less with negative numbers. However, later Arabic and European mathematicians were not (they were comfortable with subtraction, but only if the result is positive). It was only after books such as Cardano’s *Ars Magna* were published that European mathematicians became comfortable with negative numbers.

Negative number, and even complex numbers, help tremendously in the solution of polynomial equations.

13. **Positional Decimal System** (and the Hindu-Arabic Numerals). Developed in India from 600 to 800. This system consists of three ideas: (i) base 10, (ii) positional notation, it only needs symbols for 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 since the power of 10 is indicated by the position, and (iii) the use of 0 as a placeholder. The Chinese almost had a positional decimal system. The Babylonians had (ii) and later (iii), but without (i). Many civilizations had (i) but not (ii) or (iii).

Although the actually symbols for 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 varied greatly depending on location and time, the numerals built out of them are called the Hindu-Arabic numerals since they were invented by Hindu mathematicians in India and popularized by the Arabic mathematicians including al-Khwarizmi in his book *Algoritmi de numero Indorum*. In Medieval Europe, Fibonacci and others advocated for its use instead of the clunkier Roman numerals. However, it took centuries before the Roman numerals were completely replaced. (In fact, we still use Roman numerals for special purposes. I have used Roman numerals several times in this study guide!).

14. **Fractions.** Four types; (i) Unit fractions used in Egypt, and even by the Greeks and some Europeans before the Renaissance. (ii) Common fractions $\frac{a}{b}$ used in China (in the Nine Chapters) and India. Common fractions then were used in the Arab world and in Europe. (iii) Sexagesimal fractions from Babylon, but popular with Greek astronomers, and used even today in astronomy, geography and in telling time — factions of an hour are given in sexagesimal by using minutes and seconds, (iv) Decimal fractions. Chinese mathematicians liked decimal fractions. (The Arabic mathematician Al-Khashi popularized decimal fractions). In the late Renaissance, they started catching on in Europe. In fact, the only type of fraction that has fallen out of use is the unit fractions (but mathematicians are still interested in some of their mathematical properties, but mainly as a curiosity).

15. **Cubic Equations.** There are three old methods for solving cubic: the geometric method of Omar Khayam, the algebraic method of Cardano (et al.), and the numerical method. For example, Fibonacci impressed the emperor with his numeric solution.

Be able to solve the equation $x^3 + 12x = 10$ using Cardano’s method. Realize that sometimes finding even simple real roots involves using the complex numbers and that Bombelli explained how to manipulate complex numbers.

16. **Perspective.** Two important perspective facts: (i) parallel lines appear to intersect in a point on the horizon, (ii) circles appear to be ellipses or other conic sections. In fact, parabolas
can appear, in perspective, to be ellipses or circles! Techniques of perspective were developed during the Renaissance.

17. **Harmonic Series.** What is it? What was Oresme’s argument for its divergence? He was ahead of his time: diverging series became mainstream much later.

18. **Spherical triangles.** Angles do not add up to 180 degrees. In fact, you can measure area by finding the sum of the angles; be able to do this (Hint: Harriot). Spherical trigonometry was an important topic in astronomy and trigonometry in the history of mathematics. In fact, trigonometry was mainly a part of astronomy. The use of trig for planar triangles came much later! By the way, spherical geometry is a simple example of non-Euclidean geometry.

19. **Planetary Motion.** Copernicus thought that planets had circular motions around the sun. Kepler, after some false starts, determined that the orbits were actually ellipses. In fact he formulated three laws for planetary motion. Know the first two laws.

   Newton was able to simplify the theory. He showed that all three of Kepler’s laws could be derived from one law: the inverse square law. What does the inverse square law say? For example, if you multiply the distance from the sun by 3, gravitational force is multiplied by 1/9.

20. **Binomial Theorem.** You can use Pascal’s triangle to help you expand \((a + b)^n\). Use this to expand \((a + b)^6\). Which Chinese mathematician explained this?

   What if \(n\) is not an integer? Which medieval mathematician thought of fractional exponents? Answer: Oresme. Which mathematician figured out how to expand \((a + b)^n\) if \(n\) is a fraction? Answer: Newton. Usually \(a = 1\) for convenience.

   Expand \((1 + b)^{1/2}\) as a series.