1. **The Mathematics of Voting**

**Concepts:** Understand the following Concepts.

*Fairness criteria.* These are criteria that we desire in a fair electoral processes. In our class we focused on four of them. Ideally, we want methods that are guaranteed to satisfy our fairness criteria. (See page 26).

*Majority vs. Plurality.* Know the difference.

*Majority candidate.* A majority candidate is one that has received more than 50 percent of the vote.

*Condorcet candidate.* This is a candidate that will win in every possible pairwise comparison between this candidate and other candidates. (See problem 33 on page 35, problem 51 on page 39)

*Condorcet vs. Majority candidate.* Every majority candidate is a Condorcet candidate, but not every Condorcet candidate is a majority candidate. Can you explain why?

*Arrow’s impossibility theorem.* (Kenneth Arrow, 1949) It is impossible to design a voting method that will always satisfy all of the fairness criteria.

*Insincere voting.* This is voting not strictly according to your true preference, but in some other way, usually to try to defeat a candidate or candidates that you do not like. For a simple example, when you prefer a third party candidate, but you do not vote for him or her in order not to “throw away your vote”. (See page 8, and problem 11 on page 31).

**Methods:** Be able to use the following methods either for determining one winner, or for finding rankings. The only ranking method you need to know is the extended ranking method. (You do not need to worry about the recursive ranking method).

- **Plurality Method.** See example 1.2 on page 7, and the discussion after Table 1-15 on page 21. Sample problems: problem 9 on page 30, problems 41a and 42a on page 37.
- **Borda Count Method.** See page 8-9, and the discussion after Table 1-16 on page 21. Sample problems: problem 17, 19 on page 32, problems 41b and 42b on page 37, problem 51 on page 39.
- **Plurality-with-Elimination Method.** See examples 1.4, 1.5, 1.6 on pages 12–14 and the discussion on page 22. Sample problems: problem 27 on page 34, problem 33 on page 35, problems 41c and 42c on page 37.
- **Pairwise comparison Method.** See page 15–19 for examples, and the discussion after Table 1-18 on page 22. Sample problems: problem 35 and 37 on page 36, problems 41d and 42d on page 37.

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Fairness Criteria: Memorize and understand the following fairness criteria. Know examples where each are violated (one for each).

- **Majority Criterion.** If there is a majority candidate, that candidate should win. (see page 6. See problem 19 on page 32 for a violation.)
- **Condorcet Criterion.** If there is a Condorcet candidate, that candidate should win. (See page 8. See example 1.2 on page 7, and example 1.3 on page 9 for violations. See problem 19 on page 32 and problem 33 on page 35 for additional violations.)
- **Monotonicity Criterion.** If candidate $X$ is about to win, but then a group of voters decide to rate $X$ higher, then $X$ should still win. (See page 15. See example 1.6 on pages 13–14 for a violation)
- **Independence-Of-Irrelevant-Alternatives Criterion.** If candidate $X$ is about to win, but then another candidate $Y$ drops out, then $X$ should still win. (See page 18. See example 1.7 on page 17 for a violation. See problem 17b on page 32)

Other:
1. For each criterion, be able to produce an election that violates that criterion. (If you need ideas, see problem 19 on page 32, problem 33 on page 35, example 1.6 on pages 13–14, and example 1.7 on page 17).
2. For each criterion, tell me which methods are guaranteed to satisfy it, and which methods can fail it. (Answers: Majority criterion: all are guaranteed except Borda Count which can fail. Condorcet criterion: all can fail except the Pairwise comparison Method. Monotonicity Criterion: all are guaranteed to satisfy it except Plurality-with-Elimination. Independence-Of-Irrelevant-Alternatives Criterion: all methods can fail.)
3. The Method of Pairwise Comparisons satisfies most criteria. What are its problems? (Answer: it can fail the Independence-Of-Irrelevant-Alternatives Criterion. Also it often results in ties. See example 1.8 on page 18.)
4. What is another problem with methods such as the plurality method? (Answer: it can lead to insincere voting. See page 8, and problem 11 on page 31).
5. If I give you a specific election, tell me if which fairness criteria have been violated (if any). (Examples: problem 19 on page 32, problem 33 on page 35, problem 17b on page 32)
6. Which method is used to select the Olympic host city? (See page 46)
7. Which method is used in San Francisco municipal elections? (See page 10)

2. Power in Weighted Voting Systems

**Concepts:** Understand the following Concepts.
- **Dictator.** There can be only one of these. What is a dictator? Why does the dictator always have 100% of the power? (see page 54, and problems 7–10 on page 73)
- **Dummy.** There can be more than one of these. What is a dummy? Why does a dummy have 0% of the power? (See page 54, and problems 7–10 on page 73)
- **Veto player.** There can be more than one of these. What must be true for a player to have veto power? Dictators are always veto player but veto players are not always dictators. Give an example of a weighted voting system with a veto player that is not a dictator. (See page 54, and problems 7–10 on page 73. Also problem 43 on page 77 might be useful.)
Weights and Quota. The quota is the amount needed to pass a motion. The weights are the number of votes each player has. Understand how expressions such as $[10 : 5, 4, 3, 3, 2]$ describe weighted voting systems. (See page 52-53, and problem 3 on page 72).

Apparent power vs. real power. Your apparent power is the percent of votes that you have. Your real power can be computed by a power index such as the Banzhaf power index or the Shapley-Shubik Power index. Give examples where the apparent power does not match the real power (Nassau County Board at one time). Give examples where the apparent power is close to the real power (Electoral College and the European Union Council of Ministers are pretty close).

Smallest and largest allowable quotas. (See page 52). Do quotas have to be more than 50% of the votes? Can they be 90% of the votes? (See page 52, and problems 5, 6 on page 73)

Banzhaf Power Index. (See pages 56-57 and problems 11, 13, 14, 15, 17, 19 on pages 73–74.)

Coalition. How many are there? $2^n - 1$. (See page 58 and problem 41a on page 77)

Critical player. (See page 55)

Shapley-Shubik Power Index. (See page 66–67 and problems 23, 25, 27, 29 33 on pages 75–76)

Sequential Coalition. How many are there? $n!$. (see page 63–65, and problem 41c on page 77)

Pivotal player. (See page 65)

Skills: Be able to compute both types of power indexes for weighted voting systems, and rule-based voting systems. (See problems 11, 13, 14, 15, 17, 19, 23, 25, 27, 29 33 on pages 73–76).

Real world examples:

European Union. Know that in the Council of Ministers the representatives from different countries have a different number of votes based on population. Know that the Banzahf Power index is close to the apparent power, so the system works fairly well. (see page 62 and 69–70)

United Nations. Know that the Security Council has 5 permanent members and 10 non-permanent members. Know that the permanent members have veto power, and have much more power than the nonpermanent members. (see page 61 and 69)

Electoral College. Know that each state has a different number of votes, and that the number of votes is equal to the total number of senators plus the total number of members of the House of Representatives that represent that state. Know that even the small states have at least 3 votes. (See page 69 and page 84)

Nassau County Board. (See page 60). Know that at one time three districts had no power, but that Banzhaf won a court case changing the weights so that power would be better distributed.

3. Chapter 3: Fair Division

Concepts: Understand the following Concepts.

Discrete vs Continuous. (See page 90).

Fair Share and Fair Division. If there are $n$ players, then every player must get at least $1/n$ of the property. Each player can value the property differently: player $X$ must get $1/n$
of the property according to the value system of player $X$. For example, if four players divide up the property, each must feel that they got at least 25%. If player $X$ gets 35% of the value (from $X$’s value system), but player $Y$ gets 28% (from $Y$’s value system), the method is still considered fair since each is at least 25%. (See page 89 and page 90. See problem 5 on page 113 and problem 7 on page 114.)

**Important skill:** Know how to value a slice (of cake, pie, or other property) from each player’s point of view. (See problems 1–4 on page 112, problems 13–14 on page 116.)

**Methods.** Be able to use all of these.

- **Divider-Chooser (continuous, 2 players)** (see pages 91–92, and problems 9, 11, 13, 14 on pages 115–116.)
- **Lone-Divider (continuous, 3+ players).** You will only need to know this for 3 players. (See pages 93–95. See problem 15 on page 117.)
- **Lone-Chooser (continuous, 3+ players).** You will only need to know this for 3 players. (See pages 97–100, and problems 25, 27, 29 on pages 120–121.)
- **Last-Diminisher (continuous)** This method works with any number of players. (See pages 100–104. See problems 33, 35, and 37 on pages 123–124.)
- **Method of Sealed Bids (discrete)** This method works with any number of players. (See pages 105–107. See problems 39 and 44 on pages 124–125.)
- **Method of Markers (discrete).** This method works with any number of players. (See pages 107–110. See problems 47 and 49 on page 126.)

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