COMPLETE SYMMETRIC POLYNOMIALS AND RELATED COMBINATORIAL SUMS

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ABSTRACT. The focus is the identities and expressions involving sums of the type $\sum_{m=0}^{l}\binom{l}{m}m^r x^m$ and the complete symmetric polynomials $h_p(x_1, x_2, ..., x_n)$ i.e. the sum of all monomials of degree p. The recursive identity for the numbers $h_p(1, 2, ..., n)$ turns out be the same as the one for $S(n+p, n)$, the Stirling numbers of second kind, thus proving the equality between these numbers. The motivation of these results lies in the applications of the Faa Di Bruno formula for finding higher-order derivatives or coffiecients of compostions of infinitely differentiable functions or formal power series, respectively (see [4] and [5]).

For integers $p \geq 0$ and $n \geq 1$, the complete symmetric polynomial or sometimes also known as the complete homogeneous product sum, of degree p in n variables is defined as

$$h_p(x_1, x_2, ..., x_n) := \sum_{1\leq i_1 \leq i_2 \leq \cdots \leq i_p \leq p} x_{i_1} x_{i_2,...,} x_{i_p}.$$

The generating function for the h_p's is given by $\left(\prod_{i=1}^{n}(1-tx_i)\right)^{-1}$. The complete symmetric polynomials and the elementary symmetric polynomials

$$\sigma_p(x_1, x_2, ..., x_n) := \sum_{1\leq i_1 < i_2 < \cdots < i_p \leq p} x_{i_1} x_{i_2,...,} x_{i_p}$$

are related by the identity

$$\frac{1}{\sum_{j=1}^{\infty}(-1)^j \sigma_p t^p} = \sum_{j=1}^{\infty} h_p t^p \ .$$

The above relation is symmetric in the sense that it remains valid if σ_p's and h_p's are interchanged. The complete symmetric polynomials occur naturally in the contexts of Simon Newcomb's problem or Prisoner's dilemma, the multipartite partitions, the magic squares, the enumerations of permutations and combinations, the arrangements on a chess board, the generating functions of partitions, and numerous other problems (see *e.g.* [2]).

Let $h_p\left(\widehat{n}\right) := h_p(0, 1, 2, ..., n)$ denote the evaluation of the complete symmetric polynomial in $n+1$ variables of degree p. Observe that $h_p\left(\widehat{n}\right) = h_p(1, 2, ..., n)$. We use the convention $0^0 = 1$ and hence $h_0\left(\widehat{0}\right) = 1$.

1991 *Mathematics Subject Classification.* 05A19, 11B73, 05E05, ,

Key words and phrases. Symmetric polynomials, combinatorial sums, Stirling numbers of second kind.

The Stirling numbers of second kind $S(n,k)$ are defined to the number of partitions of set of n elements into k nonempty pairwise disjoint subsets. The numbers $S(n,k)$ are characterized by the recursion formula

$$\begin{aligned} S(0,0) &= 1 \\ S(n,k) &= kS(n-1,k) + S(n-1,k-1). \end{aligned}$$

We will prove the equality $h_p(\widehat{n}) = S(n+p,n)$ by showing that both of these numbers satisfy the same recursive formula. Then connections between the complete symmetric polynomials and the certain combinatorial sums are also established. These results were obtained in the process of answering certain questions that arose during author's work ([4], [5]) on infinitely differentiable functions. Indeed, the Bell polynomial version (due to J. Riordan) of Faa Di Bruno's formula for computing higher-order derivative of the composition of two infinitely differentiable functions reads

$$(f \circ g)^{(n)} = \sum_{k=0}^{n} f^{(k)}(g(t)) B_{n,k}\left(g'(t), g''(t), \cdots, g^{(n)}(t)\right), \tag{0.1}$$

where

$$B_{n,k}(x_1, x_2, \cdots, x_n) := n! \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_n=k \\ \alpha_1+2\alpha_2+\cdots+n\alpha_n=n}} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{1!^{\alpha_1} 2!^{\alpha_2} \cdots n!^{\alpha_n} \alpha_1! \alpha_2! \cdots \alpha_n!} \tag{0.2}$$

are known as Bell polynomials. A trivial modification of (0.1) yields a formula for computing coefficients of composition of formal power series.

By Theorem 1 below, it follows that

$$h_{n-p}(\widehat{n}) = B_{n,k}(1, 1, \cdots, 1) = S(n,k),$$

where the second equality is a well-known expression for the Stirling numbers of second kind.

Thus, in what follows we will use notations $h_{n-p}(\widehat{n})$ and $S(n,k)$ for Stirling numbers of second kind interchangeably.

Theorem 1. *The identity*

$$h_p\left(\widehat{j+1}\right) = \sum_{\nu=j}^{j+p} \binom{j+p}{\nu} h_{\nu-j}(\widehat{j}) \tag{0.3}$$

holds for all integers $j \geq 0$ and $p \geq 0$. In particular, $h_p(\widehat{n}) = S(n+p,n)$.

Proof. We will use the induction on p. The case $p = 0$ is trivial. Let $p \geq 1$ and assume that (0.3) holds for all complete symmetric polynomials of degree $< p$. From the definition of h_p it follows that

$$h_p(\widehat{n}) = n h_{p-1}(\widehat{n}) + h_p(\widehat{n-1}), \quad \forall n \geq 1, \forall p \geq 0. \tag{0.4}$$

By induction and by using the Pascal's identity $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$ and (0.4) repeatedly, we have

$$\begin{aligned}
&\sum_{\nu=j}^{j+p} \binom{j+p}{\nu} h_{\nu-j}(\widehat{j}) \\
= \quad & \sum_{\nu=j}^{j+p-1} \binom{j+p-1}{\nu} h_{\nu-j}(\widehat{j}) + \sum_{\nu=j}^{j+p-1} \binom{j+p-1}{\nu-1} h_{\nu-j}(\widehat{j}) \\
= \quad & h_{p-1}(\widehat{j+1}) + j \sum_{\nu=j+1}^{j+p-1} \binom{j+p-1}{\nu-1} h_{\nu-j-1}(\widehat{j}) \\
& + \sum_{\nu=j}^{j+p-1} \binom{j+p-1}{\nu-1} h_{\nu-j}(\widehat{j-1}) \\
= \quad & h_{p-1}(\widehat{j+1}) + j \sum_{\nu'=j}^{j+p-1} \binom{j+p-1}{\nu\prime} h_{\nu'-j}(\widehat{j}) \\
& + \sum_{\nu'=j-1}^{(j-1)+p} \binom{(j-1)+p}{\nu'} h_{\nu'-(j-1)}(\widehat{j-1}) \\
= \quad & h_{p-1}(\widehat{j+1}) + j h_{p-1}(\widehat{j+1}) + h_p(\widehat{j}) = h_p(\widehat{j+1}).
\end{aligned}$$

□

For a real number r and an integer $l \geq 0$, define the sum

$$P_r(l;x) := (x+1)^{-l} \sum_{m=0}^{l} \binom{l}{m} m^r x^m.$$

Observe that $P_0(l;x) = 1$ and $P_r(l,0) \equiv 1$.
The analog of Theorem 1 also holds for the functions $P_r(l;x)$.

Theorem 2. *The identity*

$$P_r(l;x) := \frac{xl}{(x+1)} \sum_{\nu=0}^{\infty} \binom{r-1}{\nu} P_\nu(l-1,x). \tag{0.5}$$

holds for all real numbers r.

Proof. By using the identity $l\binom{l-1}{m-1} = \binom{l}{m} m$, we have

$$\begin{aligned}
P_r(l;x) \quad &= \quad \frac{l}{(x+1)^l} \sum_{m=1}^{l} \binom{l-1}{m-1} m^{r-1} x^m \\
&= \quad \frac{l}{(x+1)^l} \sum_{m=0}^{l-1} \binom{l-1}{m} (m+1)^{r-1} x^{m+1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{xl}{(x+1)^l}\sum_{m=0}^{l-1}\binom{l-1}{m}x^m\sum_{\nu=0}^{\infty}\binom{r-1}{\nu}m^\nu \\
&= \frac{xl}{(x+1)^l}\sum_{\nu=0}^{\infty}\binom{r-1}{\nu}\sum_{m=0}^{l-1}\binom{l-1}{m}m^\nu x^m \\
&= \frac{xl}{(x+1)^l}\sum_{\nu=0}^{\infty}\binom{r-1}{\nu}(x+1)^{l-1}P_\nu(l-1,x) \\
&= \frac{xl}{x+1}\sum_{\nu=0}^{\infty}\binom{r-1}{\nu}P_\nu(l-1,x).
\end{aligned}$$

□

Corollary 1. *When r is a positive integer, the following recursive relation holds,*

$$P_r(l;x) := \frac{xl}{(x+1)}\sum_{\nu=0}^{r-1}\binom{r-1}{\nu}P_\nu(l-1,x). \tag{0.6}$$

In particular, by induction on r, $P_r(l;x)$ is a polynomial in l.

Example 1. We can directly compute $P_1(l;x)$ as follows.

$$\begin{aligned}
P_1(l;x) &= l\,(x+1)^{-l}\sum_{m=1}^{l}\binom{l-1}{m-1}x^m \\
&= lx\,(x+1)^{-l}\sum_{m=1}^{l-1}\binom{l-1}{m-1}x^{m-1} = \frac{lx}{(x+1)}.
\end{aligned} \tag{0.7}$$

By (0.6), we have

$$P_2(l;x) = \frac{lx}{(x+1)}\left[1+\frac{x(l-1)}{(x+1)}\right] = \frac{lx(lx+1)}{(x+1)^2}.$$

Proposition 1. *For all integers $r \geq 1$, we have*

$$P_r(l;x) := \sum_{j=0}^{r-1}\frac{C_{j,r}(x)}{(x+1)^{j+1}}l(l-1)(l-2)\cdots(l-j), \tag{0.8}$$

where $C_{j,r}$ are recursively defined as follows

$$C_{0,0}(x) = 1$$

$$C_{j,r}(x) = x\sum_{\nu=j}^{r-1}\binom{r-1}{\nu}C_{j-1,\nu}(x), \quad \forall j < r \tag{0.9}$$

Proof. By (0.7), (0.8) holds for $r = 1$. Suppose (0.8) holds for all integers $\nu < r$. By induction and by Corollary 1, we have

$$\begin{aligned}
P_r(l;x) &= \frac{xl}{(x+1)}\left[1+\sum_{\nu=1}^{r-1}\binom{r-1}{\nu}P_\nu(l-1,x)\right] \\
&= \frac{xl}{(x+1)}+\sum_{\nu=1}^{r-1}\binom{r-1}{\nu}x\sum_{j=0}^{\nu-1}\frac{C_{j,\nu}(x)}{(x+1)^{j+2}}l(l-1)(l-2)\cdots(l-j-1) \\
&= \frac{xl}{(x+1)}+\sum_{j'=1}^{r-1}x\sum_{\nu=j'}^{r-1}\binom{r-1}{\nu}\frac{C_{j'-1,\nu}(x)}{(x+1)^{j'+1}}l(l-1)(l-2)\cdots(l-j') \\
&= \frac{xl}{(x+1)}+\sum_{j'=1}^{r-1}\frac{C_{j',r}(x)}{(x+1)^{j'+1}}l(l-1)(l-2)\cdots(l-j') \\
&= \sum_{j=0}^{r-1}\frac{C_{j,r}(x)}{(x+1)^{j+1}}l(l-1)(l-2)\cdots(l-j).
\end{aligned}$$

□

Theorem 3. *The following identity*

$$P_k(l;x)=\sum_{j=0}^{k}\binom{l}{j}\frac{j!x^j}{(x+1)^j}S(k,j) \tag{0.10}$$

holds for all integers $k \geq 0$.

Proof. We can rewrite (0.8) as

$$P_k(l;x) := \sum_{j=1}^{k}\binom{l}{j}j!\frac{C_{j-1,k}(x)}{(x+1)^j}.$$

By Proposition 1, it is enough to prove that $C_{j-1,k}(\ x) = x^j h_{k-j}(\widehat{j})$. The case $k = 0$ is trivial and since $C_{0,1}(x) = x$ by (0.9), the theorem also holds for $k = 1$. Suppose $k > 1$ and $C_{j-1,\nu}(\ x) = x^j h_{\nu-j}(\widehat{j})$ for all $\nu, 1 \leq \nu < k$, and all $j \leq \nu$. By (0.9) and by Theorem 1, we have

$$C_{j-1,k}(x) = x\sum_{\nu=j}^{k-1}\binom{k-1}{\nu}C_{j-1,\nu}(x)$$

$$=\sum_{\nu=j}^{k-1}\binom{k-1}{\nu}x^j h_{\nu-j}(\widehat{j}) = x^{j+1}\cdot h_{k-j}(\widehat{j}).$$

□

Remark 1. The link between $h_p(1,2,...,n)$ and $S(n,k)$ and the following direct, induction-free proof of the identity (0.10) came to author's attention during the revision of this paper. The proof uses the well known identity

$$m^k = \sum_{j=0}^{m}\frac{m!}{(m-j)!}S(k,j),$$

which holds since both sides count the number of functions $f : \{1, 2, ..., k\} \to \{1, 2, ..., m\}$. Thus,

$$\begin{aligned}
P_k(l;x) \quad : \quad &= (x+1)^{-l}\sum_{m=0}^{l}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\binom{l}{m}S(k,j)x^m \\
&= (x+1)^{-l}\sum_{j=0}^{l}\sum_{j=m}^{l}\frac{l!}{(l-j)!}S(k,j)x^j\sum_{m=j}^{l}\binom{l-j}{m-j}x^{m-j} \\
&= (x+1)^{-l}\sum_{j=0}^{l}\frac{l!}{(l-j)!}S(k,j)x^j\,(1+x)^{l-j} \\
&= \sum_{j=0}^{l}\frac{l!}{(l-j)!}S(k,j)x^j\,(1+x)^{-j}.
\end{aligned}$$

We conclude the note with an explicit polynomial expression for $P_k(l;x)$ in terms of $h_{k-j}(\widehat{j}) = S(k,j)$.

Corollary 2. *Let*

$$A_{\nu,k} = \sum_{j=\nu}^{k}(-1)^j\sigma_{j-\nu}(\widehat{j-1})\frac{x^j}{(x+1)^j}S(k,j), 0 \le \nu \le k,$$

where $\sigma_{j-\nu}(\widehat{j-1}) = \sigma_{j-\nu}(0,1,2,...,j-1)$ *is the elementary symmetric polynomial of degree* $j-\nu$ *in* j *variables. Then*

$$P_k(l;x) = \sum_{\nu=0}^{k}(-1)^{\nu}A_{\nu,k}l^{\nu}, \tag{0.11}$$

holds for any $k \ge 0$.

Proof. Observe that for all $j \ge 1$,

$$\frac{l!}{(l-j)!} = \sum_{\nu=0}^{j}(-1)^{j-\nu}\sigma_{j-\nu}(\widehat{j-1})l^{\nu}.$$

By substituting the above expression into the identity in Theorem 3, we have

$$\begin{aligned}
P_k(l;x) \quad &= \quad \sum_{j=1}^{k}\sum_{\nu=0}^{j}(-1)^{j-\nu}\sigma_{j-\nu}(\widehat{j-1})l^{\nu}\frac{x^j}{(x+1)^j}h_{k-j}(\widehat{j}) \\
&= \quad \sum_{\nu=0}^{k}\sum_{j=\nu}^{k}(-1)^{j-\nu}\sigma_{j-\nu}(\widehat{j-1})\frac{x^j}{(x+1)^j}h_{k-j}(\widehat{j})l^{\nu},
\end{aligned}$$

which is precisely (0.11). □

Remark 2. The multivariate analogs of the Faa Di Bruno formula (0.1) for vector-valued functions (and power series) in several variables are also known (see [1] for the formula in full generality and its stochastic applications). Using these formulas one can define the multivariate Bell polynomials $B_{n,k}$ and the multivariate Stirling numbers $S(n,k)$ but not much is known beyond the definitions. Although, a future paper on the properties of multivariate $S(n,k)$ is announced in [1] but, to the best of author's knowledge, no such paper has appeared. Nevertheless, using only the definitions of $B_{n,k}$ and $S(n,k)$, it should not be difficult to extend the results in the present paper to the multivariate setting.

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