QUADRILATERALS

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This handout concerns properties of quadrilaterals in neutral geometry, especial those most useful for studying the parallel postulate. So we assume the axioms, definitions, and previously proved results of neutral geometry including the Saccheri-Legendre theorem. However, some of the definitions and results in this handout are so basic that they could have arisen in earlier geometries such as Incidence-Betweenness Geometry or IBC Geometry.

Definition 1 (Quadrilaterals)**.** To specify a quadrilateral we need to specify four (distinct) points A, B, C, D such that no three are collinear. The *quadrilateral* $\square ABCD$ is defined to be $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$. The points A, B, C, D are called *vertices*. The segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are called *sides*. The segments $\overline{AC}$ and $\overline{BD}$ are called *diagonals*. The four *vertex angles* are defined to be $\angle A = \angle DAB$, $\angle B = \angle ABC$, $\angle C = \angle BCD$, and $\angle D = \angle CDA$.

The sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides*; likewise, the sides $\overline{BC}$ and $\overline{DA}$ are opposite. The vertices A and C are called *opposite vertices*; likewise, the vertices B and D are opposite.

Remark. The above definition could have been made in Incidence-Betweenness Geometry.

1. Regular Quadrilaterals

Usually we want to consider quadrilaterals that are well-behaved. For example, we don't usually want opposite sides to intersect, and we often want the vertex angles to be such that opposite vertices are interior to these angles. This motivates the following definition.

Definition 2 (Regular Quadrilaterals)**.** A quadrilateral $\square ABCD$ is called *regular* if (i) C is interior to $\angle A$, (ii) D is interior to $\angle B$, (iii) A is interior to $\angle C$, and (iv) B is interior to $\angle D$.

The following lemma shows that in regular quadrilaterals, opposite sides do not intersect.

Lemma 1. *A quadrilateral $\square ABCD$ is regular if and only if the following holds (i) $\overline{AB}$ does not intersect $\overleftrightarrow{CD}$, (ii) $\overline{BC}$ does not intersect $\overleftrightarrow{DA}$, (iii) $\overline{CD}$ does not intersect $\overleftrightarrow{AB}$, and (iv) $\overline{DA}$ does not intersect $\overleftrightarrow{BC}$.*

Proof. This follows directly from the definition of the interior of an angle. For example, suppose that C is interior to $\angle A = \angle DAB$. Then $C \sim_l D$ with $l = \overleftrightarrow{AB}$ and and $C \sim_m B$ with $m = \overleftrightarrow{AD}$. Thus $\overline{CD}$ does not intersect $\overleftrightarrow{AB}$, and $\overline{CB}$ does not intersect $\overleftrightarrow{AD}$. Using such definitions yields the above result. $\square$

There is a corollary to the above lemma: you only have to check for regularity with one pair of opposite vertices (the other pair follows along).

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Corollary 2. *Let $\square ABCD$ be a quadrilateral. If (i) C is interior to $\angle A$, and (ii) A is interior to $\angle C$, then $\square ABCD$ is regular.*

Proof. Use the definition of the interior of angles. Check that the conditions of the lemma are satisfied. $\square$

There is one class of quadrilaterals that are obviously regular:

Definition 3 (Parallelograms)**.** A *parallelogram* is a quadrilateral such that each pair of opposite sides is parallel.

Proposition 3. *Parallelograms are regular.*

Proof. This follows from Lemma 1: since opposite sides are parallel, we do not have to worry about intersection. $\square$

The following strengthens Lemma 1.

Lemma 4. *If any three of the four conditions holds in Lemma 1, then $\square ABCD$ is regular.*

Proof. Without loss of generality, we can suppose that (i), (ii), and (iii) hold. Suppose (iv) fails. Let E be the point of intersection of $\overline{DA}$ and $\overleftrightarrow{BC}$. Since $\overline{BC}$ does not intersect $\overleftrightarrow{DA}$, we know that B and C are on the same side of $\overleftrightarrow{DA}$. So either $E * B * C$ or $E * C * B$.

First suppose that $E*B*C$. By the Crossbar Betweenness Proposition, the point C is not in the interior of $\angle EAB$. But $\angle EAB = \angle A$, so C is not in the interior of $\angle A$. But conditions (ii) and (iii) of Lemma 1, which we are assuming hold true, imply that C is interior to $\angle A$. This gives a contradiction.

If $E * C * B$ we get that B is not interior to $\angle D$, which contradictions conditions (i) and (ii). So in either case, we get a contradiction. $\square$

Corollary 5. *If any two of the four requirements of Definition 2 hold then $\square ABCD$ is regular.*

The following says that trapezoids (defined using the American convention, as opposed to the British convention) are regular.

Lemma 6. *Suppose that $\square ABCD$ has the property that (i) $\overline{AD} \parallel \overline{BC}$ and (ii) C and D are on the same side of $\overleftrightarrow{AB}$. Then $\square ABCD$ is regular.*

Proof. Hint: use Lemma 4. $\square$

Remark. All the definitions and results of this section could have been made in Incidence-Betweenness Geometry.

2. Defects of Quadrilaterals

I assume that you know the definition of the defect of a triangle. Here we extend this idea to regular quadrilaterals.

Definition 4 (Defect)**.** The *defect* $\delta ABCD$ of the quadrilateral $\square ABCD$ is defined by the formula

$$\delta ABCD \stackrel{\text{def}}{=} 360 - |\angle A| - |\angle B| - |\angle C| - |\angle D|.$$

Proposition 7. *If $\square ABCD$ is a regular quadrilateral, then $\delta ABCD = \delta ABC + \delta ADC$ and $\delta ABCD \geq 0$.*

Proof. Since $\square ABCD$ is regular, it follows that C is interior to $\angle A$ and A is interior to $\angle C$. Thus, by the Angle Measure Theorem (Neutral Geometry Handout),

$$|\angle A| = |\angle BAC| + |\angle CAD| \quad \text{and} \quad |\angle C| = |\angle BCA| + |\angle ACD|.$$

Thus

$$\begin{aligned}
\delta ABCD &= 360 - |\angle A| - |\angle B| - |\angle C| - |\angle D| \\
&= 360 - |\angle BAC| - |\angle CAD| - |\angle B| - |\angle BCA| - |\angle ACD| - |\angle D| \\
&= \Big(180 - |\angle BAC| - |\angle B| - |\angle BCA|\Big) \\
&\qquad + \Big(180 - |\angle CAD| - |\angle ACD| - |\angle D|\Big) \\
&= \delta ABC + \delta ADC.
\end{aligned}$$

Now, by the Saccheri-Legendre theorem, $\delta ABC \geq 0$ and $\delta ADC \geq 0$. Thus $\delta ABCD = \delta ABC + \delta ADC \geq 0$. $\square$

Corollary 8. *If $\square ABCD$ is a regular quadrilateral then*

$$|\angle A| + |\angle B| + |\angle C| + |\angle D| \leq 360.$$

Remark. Of course, in Euclidean geometry the defect is zero and the angle sum of a regular quadrilateral is 360. We will prove this in a later handout.

3. Rectangles

Definition 5 (Rectangles)**.** A quadrilateral $\square ABCD$ is called a *rectangle* if each of its four vertex angles is a right angle.

Lemma 9. *Rectangles are parallelograms. Hence they are regular.*

Proof. Use the Alternating Interior Angle Theorem and Proposition 3. $\square$

As we will see, there might not be any rectangles. We now study the properties that they would have if they did exist.[1]

Proposition 10. *If $\square ABCD$ is a rectangle, then $\delta ABCD = 0$.*

Proof. Hint: right angles have angle measure 90. $\square$

Corollary 11. *If $\square ABCD$ is a rectangle, then $\triangle ABC$ is a right triangle with $\delta ABC = 0$. Likewise, $\triangle ADC$ is a right triangle with $\delta ADC = 0$. Thus if rectangles exist, then there exist triangles with defect zero.*

Proof. Hint: use Proposition 7. $\square$

Proposition 12. *Suppose $\square ABCD$ is a rectangle. Then $\triangle BCA \cong \triangle DAC$.*

[1] As we will see, they exist in Euclidean Geometry, but not in Hyperbolic Geometry

Proof. Since $\delta ABC = 0$, we know that $|\angle BAC| + |\angle BCA| = 90$. But since C is interior to $\angle BAD$ (because rectangle are regular), we get

$$|\angle BAC| + |\angle CAD| = |\angle BAD| = 90.$$

Set the two equations equal and solve: $|\angle BCA| = |\angle CAD|$. By AAS, we get

$$\triangle BCA \cong \triangle DAC.$$

□

Corollary 13. *Opposite sides of a rectangle are congruent.*

Proof. Let $\square ABCD$ be a rectangle. By the previous proposition, $\triangle BCA \cong \triangle DAC$. So $\overline{BC} \cong \overline{DA}$ and $\overline{AB} \cong \overline{CD}$. □

4. Stacking Rectangles

Rectangles can be "stacked" to form larger and larger rectangles. This fact is important in the proof that if rectangles exist then all triangles have defect zero (this proof will be given in a future handout).

Proposition 14. *Suppose there is a rectangle whose sides have length x and y. Then there is a rectangle with sides of length $2x$ and y.*

Proof. Let $\square ABCD$ is a rectangle with $x = \left|\overline{AB}\right| = \left|\overline{CD}\right|$ and $y = \left|\overline{BC}\right| = \left|\overline{AD}\right|$. Let E be a point such that $E * A * B$ and $\overline{EA} \cong \overline{AB}$. Thus $\left|\overline{EB}\right| = 2x$. Likewise, let F be a point such that $F * D * C$ and $\overline{FD} \cong \overline{CD}$. Thus $\left|\overline{FC}\right| = 2x$. Our goal is to show that $\square EBCF$ is a rectangle.

Observe that $\angle EAD$ and $\angle FDA$ are right since they are supplementary to angles of a rectangle. By SAS, $\triangle EAD \cong \triangle BAD$. So $\overline{ED} \cong \overline{BD}$ and $\angle EDA \cong \angle BDA$. Now E is interior to $\angle FDA$ (we leave this to the reader), so $|\angle FDE| = 90 - |\angle EDA|$. Since $\angle EDA \cong \angle BDA$, we have $|\angle FDE| = 90 - |\angle BDA|$. Since $\triangle BAD$ is a right triangle of defect 0, we have $|\angle ABD| = 90 - |\angle BDA|$. Therefore, $|\angle FDE| = |\angle ABD|$.

So $\angle FDE \cong \angle ABD$ and $\overline{FD} \cong \overline{AB}$ and $\overline{ED} \cong \overline{BD}$. Thus $\triangle FDE \cong \triangle ABD$ by SAS. In particular, $\angle F$ is right. A similar argument shows $\angle E$ is right. Thus $\square EBCF$ is a rectangle. □

Exercise 1. Show that E is interior to $\angle FDA$ in the above proof.

Proposition 15. *Suppose there is a rectangle. Then there are arbitrarily large rectangles in the following sense. If M is any (large) real number then there is a rectangle whose sides all have length bigger than M.*

Proof. Let $\square ABCD$ is the given rectangle. Let $x = \left|\overline{AB}\right| = \left|\overline{CD}\right|$ and $y = \left|\overline{BC}\right| = \left|\overline{AD}\right|$. By the above proposition, there is a rectangle with sides $2x$ and y. Now apply the proposition again and get a rectangle with sides 2^2x and y. One can keep doubling until one gets a rectangle with sides 2^kx and y, where k is chosen large enough so that $2^kx > M$.

This gives a rectangle with sides y and 2^kx. The above proposition gives a rectangle with sides $2y$ and 2^kx. By repeating, we can keep doubling until we get a rectangle with sides 2^ly and 2^kx where l is chosen so that $2^l > M$. □

5. Saccheri Quadrilaterals and Lambert Quadrilaterals

Since rectangles might not exist, we study the next best thing: Saccheri quadrilaterals and Lambert quadrilaterals.

Definition 6 (Saccheri Quadrilateral)**.** A *Saccheri quadrilateral* $\square ABCD$ is a quadrilateral such that (i) $\angle B$ and $\angle C$ are right, (i) $\overline{AB} \cong \overline{CD}$, and (iii) A and D are on the same side of $\overleftrightarrow{BC}$. The angles $\angle B$ and $\angle C$ are called *base angles*, and the side $\overline{BC}$ is called *base*. We call $\overline{AB}$ and $\overline{CD}$ the *sides*.

Exercise 2. Show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel in the above definition. Hint: use the right angles.

Lemma 16. *Saccheri quadrilaterals $\square ABCD$ are regular quadrilaterals.*

Proof. This follows from Lemma 6. $\square$

Exercise 3. Show that if x and y are two real numbers, there is a Saccheri quadrilateral with base of length x and sides of length y.

The following is an important result concerning Saccheri quadrilaterals.

Proposition 17. *Let $\square ABCD$ be a Saccheri quadrilateral with base angles $\angle B$ and $\angle C$. Then $\angle A$ and $\angle D$ are congruent to each other and are acute or right. (If they are right, then $\square ABCD$ is also a rectangle.)*

Proof. First observe that $\triangle ABC \cong \triangle DCB$ by SAS. So $\overline{AC} \cong \overline{BD}$. Thus $\triangle BAD \cong \triangle CDA$ by SSS. So $\angle A \cong \angle D$.

Thus $\delta ABCD = 360 - 90 - 90 - |\angle A| - |\angle C| = 180 - 2|\angle A|$. Since $\delta ABCD \geq 0$ for all regular quadrilaterals, we get $|\angle A| \leq 90$ for Saccheri quadrilaterals. $\square$

Definition 7 (Lambert Quadrilateral)**.** A *Lambert quadrilateral* is quadrilateral with at least three right vertex angles.

Lemma 18. *Lambert quadrilaterals are parallelograms. Thus they are regular quadrilaterals.*

Proof. Hint: Alternating Interior Angle Theorem. $\square$

Proposition 19. *Let $\square ABCD$ be a Lambert quadrilateral with angles $\angle B$, $\angle C$, and $\angle D$ all right. Then $\angle A$ is acute or right.*

Exercise 4. Prove the above theorem

The main result concerning Lambert quadrilaterals is the following:

Proposition 20. *Let $\square ABCD$ be a Lambert quadrilateral with angles $\angle B$, $\angle C$, and $\angle D$ all right, but with $\angle A$ acute. Then $\overline{AB} > \overline{CD}$ and $\overline{DA} > \overline{BC}$.*

Proof. We will prove $\overline{AB} > \overline{CD}$; proving $\overline{DA} > \overline{BC}$ is similar. Suppose otherwise. Then either $\overline{AB} \cong \overline{CD}$ or $\overline{AB} < \overline{CD}$.

Suppose first that $\overline{AB} \cong \overline{CD}$. Then $\square ABCD$ is a Saccheri quadrilateral. Thus $\angle A \cong \angle D$. So $\angle A$ is right, a contradiction.

Suppose $\overline{AB} < \overline{CD}$. Let E be a point with $C * E * D$ and $\overline{CE} \cong \overline{AB}$. Then $\square ABCE$ is a Saccheri quadrilateral. Thus $\angle EAB \cong \angle AEC$. By the Exterior Angle Theorem,

$\angle AEC > \angle D$, but $\angle D > \angle A$ since $\angle A = \angle BAD$ is acute. So $\angle AEC > \angle BAD$ by transitivity. Now E is in the interior of $\angle BAD$, so $\angle BAE < \angle BAD$. By transitivity, $\angle AEC > \angle BAE$. This contradicts the earlier observation that $\angle EAB \cong \angle AEC$.

So in either case, we get a contradiction. □

Exercise 5. Show, in the above proof, that E is in the interior of $\angle BAD$. Hint: use parallelism and the definition of interior.

From a Saccheri quadrilateral, we can get a Lambert quadrilateral by choosing midpoints (which exist by an earlier result). This will yield some important corollaries.

Proposition 21. *Let $\square ABCD$ be a Saccheri quadrilateral with base angles $\angle B$ and $\angle C$. Let M be the midpoint of $\overline{AD}$ and let N be the midpoint of $\overline{BC}$. Then $\angle AMN$ and $\angle BNM$ are right. In particular, $\square ABNM$ and $\square MNCD$ are Lambert quadrilaterals.*

Proof. By SAS, $\triangle ABN \cong \triangle DCN$. So $\overline{AN} \cong \overline{DN}$. By SSS, $\triangle AMN \cong \triangle DMN$. Thus $\angle AMN \cong \angle DMN$. This tells that $\angle AMN$ is right since it is congruent to its supplementary angle.

By Proposition 17, $\angle A \cong \angle D$. By SAS, $\triangle BAM \cong \triangle CDM$. So $\overline{BM} \cong \overline{CM}$. So $\triangle BMN \cong \triangle CMN$ by SSS. Thus $\angle BNM \cong \angle CNM$. This tells that $\angle BNM$ is right since it is congruent to its supplementary angle. □

Corollary 22. *Let $\square ABCD$ be a Saccheri quadrilateral with base angles $\angle B$ and $\angle C$. Assume that $\square ABCD$ is not a rectangle. Then $\overline{AD} > \overline{BC}$.*

Proof. Since $\square ABCD$ is not a rectangle, $\angle A$ is acute. Let M and N be as in the above proposition. So $\square ABNM$ is a Lambert quadrilateral. By an earlier property of Lambert quadrilateral $\overline{AM} > \overline{BN}$. Thus $\overline{AD} > \overline{BC}$. □

Corollary 23. *All Saccheri quadrilaterals are parallelograms.*

Proof. Let $\square ABCD$ be a Saccheri quadrilateral with base angles $\angle B$ and $\angle C$. Let M and N be as in the above proposition. Then $\overleftrightarrow{MN}$ is perpendicular to both $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. So $\overleftrightarrow{AD} || \overleftrightarrow{BC}$.

Since $\overleftrightarrow{BC}$ is perpendicular to both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. So $\overleftrightarrow{AB} || \overleftrightarrow{CD}$. □

6. A Lemma

The following will be useful later.

Lemma 24. *Let $\square ABCD$ be a quadrilateral with right angles $\angle B$ and $\angle C$, and such that A and D are on the same side of $\overleftrightarrow{BC}$. Then $\overline{AB} > \overline{DC}$ implies that $\angle A < \angle D$.*

Proof. Let A' be a point such that $A * A' * B$ and $\overline{A'B} \cong \overline{DC}$. Then $\square A'BCD$ is a Saccheri quadrilateral. So, by Proposition 17, $\angle BA'D \cong \angle A'DC$.

But $\angle BA'D > \angle A$ by the Exterior Angle Theorem. So $\angle A'DC > \angle A$ by substitution. Also, $\angle ADC > \angle A'DC$ since A' is interior to $\angle ADC$.[2] Thus $\angle A < \angle ADC = \angle D$. □

Dr. Wayne Aitken, Cal. State, San Marcos, CA 92096, USA
E-mail address: `waitken@csusm.edu`

[2]Can you see why A' is interior? Hint: show that $\square ABCD$ is regular, so that B is interior. Now show A' is also interior by referring to the definition and the fact that $A * A' * B$.