STUDY GUIDE FOR TEST 1

MATH 410. SPRING 2006. INSTRUCTOR: PROFESSOR AITKEN

1. Incidence-Betweenness Geometry

A large part of the test will be about Incidence-Betweenness Geometry. Because it is so important, I wrote a detailed ten page handout on this geometry. Incidence Geometry is also discuss in Chapter 2 of the textbook, and Incidence-Betweenness Geometry is discussed in the first part of Chapter 3.

You should memorize all the axioms and their names (I-1, I-2, I-3, B-1, B-2, B-3, B-4). You should memorize all definitions.

You should know all the propositions (but you do not need to memorize their numbers). You should be able to prove any of them (except perhaps Proposition 28; since Proposition 28 was not covered in class, I will make its proof extra-credit only). You should know what logical order that the propositions were proved. In other words, if I ask you to prove a proposition, you need to know what propositions were proved before it, because those can be used in the proof. (Note: the terms "proposition" and "theorem" are used interchangeably in this unit. But the term "theorem" is sometimes reserved for deeper propositions.)

2. Models

You need to know what a model is. If I give you an interpretation, then you need to tell me what axioms it is a model for. You should be able to answer other questions such as "does this model satisfy the hyperbolic parallel property?"

If a model interprets "line", "point", and "incidence", and if all three of the incidence axioms are true in the model, then the model is a model for Incidence Geometry. If a model interprets "line", "point", "incidence", and "betweenness" and if all seven of the incidence and betweenness axioms are true in the model, then the model is a model for Incidence-Betweenness Geometry.

You should know this basic fact: *every theorem of a type of geometry is true of every model for that type of geometry* since such theorems logically follow from axioms that are true in such models. For example, you can prove in incidence geometry that for every point there is at least one line not passing through it. So every model of incidence geometry has that feature. If a model fails that, it must fail one or more of the axioms of incidence geometry.

Conversely however, if a statement is true in a model for a type of geometry, it might not be a theorem for that type of geometry. For example, there are models of incidence geometry that are finite, but you cannot prove that there is a finite number of points in incidence geometry. (There are many models of incidence geometry that have an infinite number of points).

These observations do not just apply to geometry, they apply to any axiom system. For example, models of group theory are called "groups". Every theorem of group theory is true

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in every model (group), but not everything that is true in a given group is a theorem of group theory.

Models are an important tool for understanding geometry. One thing models can do is show *unprovability*. More specifically, *if a statement fails in some model for an axiom system, then by the above observation, that statement cannot be proved*. This allows us to stop trying to prove a theorem that is impossible to prove. Most interesting is the case where there is a model where a statement S holds and another where its negation $\sim S$ holds. In that case, neither S or $\sim S$ is provable in the axiom system. In other words, S is undecidable in the axiom system.

Models are also useful to show *independence* of axioms. We say that n axioms are *independent* when no axiom can be proved from the other $n-1$ axioms. To show n axioms are independent, you just need to find models: for each axiom you need to find a model where that axiom is false, but the other $n-1$ are true.

Models can be used to show that an axiom system does not lead to a contradiction. There are axiom systems in which a contradiction $S \wedge \sim S$ can be proved. These are axiom systems about nothing, since there are no models for such axiom systems. If it is possible to prove a contradiction in an axiom system, then we say that the axiom system is *inconsistent*. Otherwise, we say that the axiom system is *consistent*. To show an axiom system is consistent, you just need to find one model for the axioms. So, even the simple three point model we developed (with three lines) is enough to show that the axioms of incidence geometry are consistent.

You should understand the idea of isomorphisms of models. Suppose that you have a statement for S in a type of geometry or axiom system (for example the hyperbolic parallel property). Then if S is true of a model, then it is true in all models isomorphic to it. In other words, isomorphic models are essentially the same. To show two models are not isomorphic, find a statement of geometry that is true of one but false in the other. To show two models are isomorphic, you need to construct one-to-one and onto functions (bijections) with certain properties.

If an axiom system has the property that all its models are isomorphic, then we say that the axioms are *categorical*. This means that you are done, you do not need to add any more axioms to specify your geometry. For example, Hilbert's axioms are categorical. The axioms we have studied so far are only a subset of Hilbert's axioms, and are not categorical. (For example, $\mathbb{R}^2$ and $\mathbb{Q}^2$ are non-isomorphic models of Incidence-Betweenness geometry.)

3. Logic

You should understand all the basic background logic needed for our geometry. In this course, we consider set theory as part of the basic background logic. (In mathematical logic, set theory is usually not considered part of basic logic, but an axiom system of its own.)

By way of contrast: from Euclid's time to about 1850, ideas about betweenness were considered part of the background logical-reasoning tools that could be used without proof or explicit statement.

4. Hilberts Axioms for Euclidean Geometry

In the late 1800s mathematicians became concerned enough about the gaps in Euclid, especially concerning existence and betweenness, to develop better axioms systems. Pasch was a major contributer. Hilbert developed a categorical axiom system for Eulidean geometry, and showed that these axioms are independent.

The axioms I-1, I-2, I-3, B-1, B-2, B-3, B-4 from class are a subset of Hilbert's axioms. For the next test, we will study the rest.

If you replace Hilbert's Euclidean Parallel Postulate with the Hyperbolic Parallel Postulate, you get categorical axioms for "hyperbolic geometry". In this class we will learn both Euclidean and hyperbolic geometry. (Hilbert's geometry without either the Euclidean or Hyperbolic Parallel Postulates is called "neutral" or "absolute" geometry. We will study this in Chapter 4.).

Just because Euclid's axioms have gaps, does not diminish the importance of his book, the *Elements*. Think of Euclid as giving a different approach: his reasoning included obvious (to him) facts about betweenness and existence, while the modern point of view is to include only basic logic and set theory in our deduction from axioms to theorems. From the modern point of view, if you want to use betweenness facts you need to include them as axioms or prove them as theorems.

5. Parallel Postulate

A major theme of our textbook is the history of the parallel postulate.

Know the Euclidean Parallel Property, the Hyperbolic Parallel Property, and the Elliptic Parallel Property. (In some situations, these properties will be assumed as postulates).

Euclidean Parallel Property. *For every line l and every point P not on l, there is exactly one line m passing through P such that $l \parallel m$.*

Hyperbolic Parallel Property. *For every line l and every point P not on l, there are at least two lines passing through P that are parallel to l.*

Elliptic Parallel Property. *For every line l and every point P not on l, there are zero lines passing through P that are parallel to l.*

This last property can be stated in a much simpler form:

Elliptic Parallel Property (Version 2)**.** *There are no parallel lines.*

Actually, Euclid's original parallel postulate was different than the Euclidean Parallel Property:[1]

Euclid's Fifth Postulate. *Suppose l and m are lines, both intersecting a third line t. Suppose that the interior angles on one side of t have degree measures adding to less than 180 degrees, then l and m intersect on that side of t.*

6. History

Know that the first (extant) geometry book with proofs is Euclid's *elements*. Know that until recently, Euclid was considered the model of deductive proof.

Know a little about the history of the parallel postulate (from Chapter 1). For example, you should know that many mathematicians were unhappy with Euclid's Fifth Postulate.

[1] Euclid did not use degrees. He said "two right angles" instead of 180 degrees. Also, he did not use the phrase "angle measure".

They felt it was too complex for an axiom, and should be a proposition instead. Greek, Arab, and European mathematicians worked on the problem of eliminating this axiom.

The first strategy was to eliminate the Fifth Postulate altogether. Several famous mathematicians before 1800 thought that they could do this, but their proofs were later found to be unsatisfying since they rested on unstated assumption that were just as powerful as Euclid's Fifth Postulate.

The second strategy was to replace the Fifth Postulate by another axiom that was in some sense simpler. This was successful. For example, the Euclidean Parallel Property is a good replacement for the original Fifth Postulate.

Know that after 1800 some mathematicians gave up trying to prove the Fifth Postulate (or its replacements), and came to the conclusion that the Fifth Postulate was independent of the other axioms of geometry. They started to believe that non-Euclidean geometry was a consistent and interesting geometry on its own, and began to prove interesting theorems about it.

We will study this history in more detail as the course progresses.

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