EUCLIDEAN GEOMETRY

MATH 410, CSUSM. SPRING 2008. PROFESSOR AITKEN

1. Introduction

Euclidean Geometry is obtained by adding a parallel axiom to Neutral Geometry. What parallel axiom do you add? It turns out there are several axioms that work equally well. For example, there is Euclid's original Postulate V, there is a postulate asserting uniqueness of parallels, and there is the converse to the alternating interior angle theorem. It turns out that these are all in some sense equivalent, and there are others as well.

A good place to do comparison shopping for parallel axioms is in a geometry that has as much as possible without actually containing a parallel axiom. This geometry is Neutral Geometry. When we say that parallel axioms A and B are equivalent, what we mean is that $A \iff B$ can be proved in Neutral Geometry. In this document we give several statements, and show that they are all equivalent in Neutral Geometry. Any statement equivalent in this way to Euclid's Fifth Postulate, or anything equivalent to Euclid's Fifth Postulate, is called a *Euclidean condition*.

We will only discuss the most straightforward Euclidean conditions in this document. Others will be discussed in a later document.

2. Equivalence of E5P and UPP

Recall Euclid's Fifth Postulate (E5P): *For any pair of distinct lines l and m, and any transversal t to these lines, if the sum of the angle measures of the two interior angles on a given side of t is less than 180, then l and m must intersect on that side of t.* We will officially phrase this as follows:

Definition 1 (E5P)**.** *Euclid's Fifth Postulate* (E5P) is the following statement: *Suppose that (i) $l = \overleftrightarrow{PA}$ and $m = \overleftrightarrow{QB}$ are distinct lines such that $P \neq Q$, (ii) A and B are on the same side of $t = \overleftrightarrow{PQ}$, and (iii) $|\angle APQ| + |\angle BQP| < 180$. Then l and m intersect in a point C, and A, B and C are all on the same side of t.*

Remark 1. We have modernized this a bit. Euclid did not use degrees, so the statement in his *Elements* does not mention 180. Instead it mentions the sum of two right angles. Also the *Elements* does not clearly distinguish between angles and angle measures.

There is another statement that many prefer to use as an axiom instead:

Definition 2 (UPP)**.** The *Unique Parallel Property* (UPP) is the following statement: *Given any line l and any point P not on l, there is a unique line containing P that is parallel to l.*

Date: Spring 2008. Version of April 17, 2008.

Of course, neither Euclid's Fifth Postulate (E5P) nor the Unique Parallel Property (UPP) is an axiom of Neutral Geometry. It turns out that neither is a theorem of Neutral Geometry. What can be proved is that these statements are equivalent in Neutral Geometry.

Proposition 1. *In Neutral Geometry E5P and UPP are equivalent. In other words,*

$$\text{E5P} \iff \text{UPP}$$

is a theorem in Neutral Geometry.

Proof. First assume E5P. We must prove UPP. So let l be a line and P a point not on l. By a result of IBC Geometry, we can drop a perpendicular from P to l. Let Q be the foot. Thus $l \perp \overleftrightarrow{PQ}$. Again, by a result of IBC Geometry there is a line m_1 perpendicular to $\overleftrightarrow{PQ}$ that contains P. According to IBC Geometry, $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$.

We must show that there are no other lines parallel to l containing P. To do so, suppose that m_2 is another such line. Then choose points X and Y on m_2 such that $X * P * Y$ (Axioms I-2 and B-2). One of the angles $\angle XPQ$ or $\angle YPQ$ must be acute since they are not right, and they are supplementary. Thus the sum of interior angles is less than 180 on one side of $\overleftrightarrow{PQ}$. So m_2 intersects l by E5P (our assumption). This contradicts the assumption that m_2 is parallel to l. Thus m_1 is the unique parallel. We have established UPP.

Now assume UPP. We must prove E5P. So, suppose that (i) $l = \overleftrightarrow{PA}$ and $m = \overleftrightarrow{QB}$ are distinct lines such that $P \neq Q$, (ii) A and B are on the same side of $t = \overleftrightarrow{PQ}$, and (iii) $|\angle APQ| + |\angle BQP| < 180$. Let D be a point such that $B * Q * D$ (Axiom B-2). Then $|\angle PQD| = 180 - |\angle PQB|$. This implies that $|\angle PQD| \neq |\angle APQ|$ (otherwise $|\angle APQ| + |\angle BQP| = 180$). By Axiom C-5, there is a ray $\overrightarrow{PX}$ such that $\angle XPQ \cong \angle PQD$ and such that X and A are on the same side of t (and so X and D are on opposite sides of t). Let $l' = \overleftrightarrow{PX}$. Then $l' \parallel m$ by the Alternating Interior Theorem of Neutral Geometry. Now $l \neq l'$ since $|\angle XPQ| \neq |\angle APQ|$. By UPP this means that l is not parallel to m. Thus l must intersect m at a point, call it C.

To establish E5P we still need to show that A and C are on the same side of t. Suppose A and C are on opposite sides of t. Then we have a triangle $\triangle CPQ$ such that $\angle CPQ$ is supplementary to $\angle APQ$, and angle $\angle CQP$ is supplementary to $\angle BQP$. Since $|\angle APQ| + |\angle BQP| < 180$, we get $|\angle CPQ| + |\angle CQP| > 180$ which contradicts an earlier theorem that the sum of any two angle measures of a triangle are less than 180. Thus C and A must be on the same side of t. We have established E5P. □

3. Equivalence of UPP and ConAIA

Recall that the Alternating Interior Angle Theorem is a theorem of IBC Geometry. We now consider the converse:

Definition 3 (ConAIA)**.** The converse to the Alternating Interior Angle Theorem is the following statement: *for any pair of distinct parallel lines l and m and any transversal t to these lines the alternating interior angles are equal.*

In other words, *if $l = \overleftrightarrow{AP}$ and $m = \overleftrightarrow{QB}$ are parallel, and if A and B are on opposite sides of $t = \overleftrightarrow{PQ}$, then $\angle APQ \cong \angle BQP$.*

Proposition 2. *In Neutral Geometry UPP and ConAIA are equivalent. In other words,*

$$UPP \iff ConAIA$$

is a theorem in Neutral Geometry.

Proof. First assume that UPP holds. We must prove ConAIA, so suppose $l = \overleftrightarrow{AP}$ and $m = \overleftrightarrow{QB}$ are parallel, and that A and B are on opposite sides of $t = \overleftrightarrow{PQ}$. We need to show that $\angle APQ \cong \angle BQP$. By Axiom C-5 there is a point A' on the same side of t as A such that $\angle A'PQ \cong \angle BQP$. By the Alternating Interior Angle Theorem of IBC Geometry, this implies that $l' \parallel m$ where $l' = \overleftrightarrow{A'P}$. By UPP (our assumption) we have that $l = l'$ since both lines contain P and both are parallel to m. So $\angle APQ = \angle A'PQ$. Thus $\angle APQ \cong \angle BQP$ as desired.

Now assume ConAIA. We must prove UPP, so let l be a line and P a point not on l. By a result of IBC Geometry, we can drop a perpendicular from P to l. Let Q be the foot. Thus $l \perp \overleftrightarrow{PQ}$. Again, by a result of IBC Geometry there is a line m_1 perpendicular to $\overleftrightarrow{PQ}$ that contains P. According to IBC Geometry, $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$. We must show that there are no other lines parallel to l containing P. To do so, suppose that m_2 is any such such line. ConAIA implies that $m_2 \perp \overleftrightarrow{PQ}$. By a result of IBC Geometry there is a unique perpendicular to $\overleftrightarrow{PQ}$ containing P. Thus $m_1 = m_2$. We have established UPP. □

4. The Proclus Property

The ancient Greek Mathematician Proclus used something like the following statement in his study of Euclid's Fifth Postulate.

Definition 4 (Proclus)**.** The *Proclus Property* is the following statement: *Suppose l, m, t are distinct lines. If $l \parallel m$ and if t intersects m, then t must intersect l.*

Proposition 3. *In Neutral Geometry UPP and the Proclus Property are equivalent. In other words,*

$$UPP \iff Proclus\ Property$$

is a theorem in Neutral Geometry.

Proof. First assume that UPP holds. We must prove the Proclus Property, so suppose that $l \parallel m$ and that a third line t intersects m. Let P be the point of intersection. Our goal is to show that t also intersects l. Suppose it didn't, then t and m would both be parallel to l and would both contain P, contradicting UPP.

Now assume the Proclus Property. We must prove UPP, so let l be a line and P a point not on l. From IBC Geometry, there is at least one parallel m to l containing P. Suppose m' is another. Observe that m' intersects m at P. So, by the Proclus Property, m' must also intersect l, a contradiction. Therefore, m is the unique parallel to l containing P. □

5. The Transitivity of Parallels Property

We now study one more property that is equivalent to UPP.

Definition 5 (TPP)**.** The *Transitivity of Parallels Property* (TPP) is the following statement: *Let l, m, n be distinct lines. If $l \parallel m$ and $m \parallel n$ then $l \parallel n$.*

Proposition 4. *In Neutral Geometry the Proclus Property and TPP are equivalent. In other words,*

$$\text{Proclus Property} \iff \text{TPP}$$

is a theorem in Neutral Geometry.

Proof. First assume the Proclus Property. We wish to show TPP, so l, m, n be distinct lines with $l \parallel m$ and $m \parallel n$. Our goals is to show $l \parallel n$. Suppose otherwise, that l and n intersect. Since l intersects n it must intersect m by the Proclus Property (because $m \parallel n$). Thus l is not parallel to m, a contradiction.

Suppose TPP. We wish to show the Proclus Property, So let l, m, t be distinct lines such that $l \parallel m$ and such that t intersects m. We must show t intersects l. Suppose not, then $t \parallel l$. By TPP, $t \parallel m$. This is a contradiction. □

Corollary 5. *In Neutral Geometry UPP and TPP are equivalent. In other words,*

$$\text{UPP} \iff \text{TPP}$$

is a theorem in Neutral Geometry.

Proof. Combine the above proposition with Proposition 3. □

6. Euclidean Geometry

Euclidean Geometry consists of 5 undefined terms, 16 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. We will adopt all the notation and definitions from Neutral Geometry.

The Primitive Term Axiom for Euclidean Geometry is a preliminary axiom telling us what type of objects all the primitive terms are supposed to represent. The Primitive Term Axiom in neutral geometry is exactly the same as for IBC geometry.

Axiom (Primitive Terms)**.** *The basic type of object is the point. Lines are sets of points. Betweenness is a three place relation of points. If P, Q, R are points, then $P * Q * R$ denotes the statement that the betweenness relation holds for (P, Q, R). Segment congruence is a two place relation of line segments, and angle congruence is a two place relation of angles. If $\overline{AB}$ and $\overline{CD}$ are line segments, then $\overline{AB} \cong \overline{CD}$ denotes the statement that the segment congruence relation holds between $\overline{AB}$ and $\overline{CD}$. If α and β are angles, then $\alpha \cong \beta$ denotes the statement that the angle congruence relation holds between α and β.*

The axioms of Euclidean Geometry include the above Primitive Term Axiom together with the axioms I-1, I-2, I-3, B-1, B-2, B-3, B-4, C-1, C-2, C-3, C-4, C-5, C-6, Dedekind's Axiom, and the following Axiom.

Axiom (UPP)**.** *Given any line l and any point P not on l, there is a unique line containing P that is parallel to l.*

Since the axioms of Neutral Geometry are a subset of the axioms of Euclidean Geometry, all the propositions of Neutral Geometry automatically hold in Euclidean Geometry.

The following can be called a *meta-lemma* since it is not really a result of Euclidean Geometry, but a result about results in Euclidean Geometry.

Lemma 6. *Suppose X is a statement, and that $UPP \implies X$ is a theorem of Neutral Geometry, then X is a theorem of Euclidean Geometry.*

Proof. In order to give a proof of X in Euclidean Geometry, first give the steps leading to $UPP \implies X$. Since every proof in Neutral Geometry is automatically a proof in Euclidean Geometry, these steps will be valid in Euclidean Geometry. Now assert UPP. This is valid in Euclidean Geometry since UPP is an axiom. In your last step, use the logical rule of *modus ponens* to conclude X. □

Corollary 7. *The statements E5P, ConAIA, the Proclus Property, and TPP are theorems of Euclidean Geometry.*

Remark 2. We will see later that none of these are theorems of Neutral Geometry, and that they are all false in Hyperbolic Geometry.

7. Euclidean Conditions

Definition 6 (Euclidean Condition)**.** A *Euclidean condition* is a statement X such that

$$UPP \iff X$$

is a theorem of Neutral Geometry.

Clearly UPP is a Euclidean Condition since $UPP \iff UPP$ is trivially a theorem. We have seen also that E5P, ConAIA, the Proclus Property, and TPP are all Euclidean conditions. Here is an important meta-proposition concerning Euclidean conditions.

Proposition 8. *Every Euclidean condition is a theorem in Euclidean Geometry.*

Proof. This follows from Lemma 6. □

We can prove something much stronger than the above. We can show that any Euclidean condition can be used as the final axiom for Euclidean Geometry. In other words, if you do not like UPP you can replace it with any other Euclidean condition:

Proposition 9. *If X is an Euclidean condition and if we replace UPP as an axiom by X, the resulting geometry will be equivalent to Euclidean Geometry. In other words, you can prove exactly the same statements in both geometries.*

Proof. Let EG be Euclidean Geometry, and let EG$'$ be the geometry you get when you replace UPP by the Euclidean condition X.

First suppose S is a statement that can be proved in EG. Then $UPP \implies S$ can be proved in Neutral Geometry: just take the proof of S and replace every reference to the axiom UPP with a reference to the hypothesis UPP. Now $X \iff UPP$ can be proved in Neutral Geometry since X is a Euclidean Condition. Since $UPP \implies S$ and $X \iff UPP$ can be proved in Neutral Geometry, we can prove $X \implies UPP$ in Neutral Geometry. This means $X \implies S$ can be proved in EG$'$ as well since EG$'$ has all the axioms of Neutral Geometry. Now X can be trivially proved in EG$'$ since it is an axiom. Thus we can prove S in EG$'$ (using *modus ponens* if necessary).

A similar argument shows that any statement that can be proved in EG$'$ can be proved in EG. Thus both geometries prove the same statements. □

Remark 3. We conclude from this that E5P and UPP work equally well as an axiom. Euclid chose to use E5P, but most modern textbooks use UPP instead.

8. Theorems of Euclidean Geometry

We end with some propositions of Euclidean Geometry. Some of these will be seen to be Euclidean conditions in later handouts. (Recall that any statement proved to be a Euclidean condition, automatically becomes a theorem in Euclidean Geometry: for example, conAIA, or TPP are theorems of Euclidean Geometry).

Proposition 10. *If $\triangle ABC$ is a triangle, then*

$$|\angle A| + |\angle B| + |\angle C| = 180.$$

Proof. Let m be the line containing A that is parallel to $\overleftrightarrow{BC}$ (exists by UPP). Let D and E be points on m such that $D * A * E$ (Axiom I-2, Axiom B-2). We can choose D and E so that $\angle DAB$ and $\angle B$ form alternating interior angles, $\angle EAC$ and $\angle C$ form alternating interior angles, and C is interior to $\angle EAB$ (using properties of Incidence-Betweenness Geometry[1]).

Since C is interior to $\angle EAB$, we have $|\angle EAB| = |\angle EAC| + |\angle A|$. Since $\angle DAB$ and $\angle EAB$ are supplementary,

$$180 = |\angle DAB| + |\angle EAB| = |\angle DAB| + |\angle EAC| + |\angle A|.$$

By ConAIA, $\angle DAB \cong \angle B$ and $\angle EAC \cong \angle C$. So

$$|\angle DAB| + |\angle EAC| + |\angle A| = |\angle B| + |\angle C| + |\angle A|.$$

The result follows. □

Proposition 11. *Let A, B, C, D be four points such that (i) A, B, C are not collinear and D is interior to $\angle ABC$, and (ii) A, D, C are not collinear and B is interior to $\angle ADB$. Then*

$$|\angle DAB| + |\angle ABC| + |\angle BCD| + |\angle CDA| = 360.$$

Remark 4. As we will see in the handout on quadrilaterals, the hypotheses of the above theorem imply that $\square ABCD$ is a regular quadrilateral. The theorem says that the sum of the angles of a regular quadrilateral is 360.

Proof. Let $\beta = |\angle ABC|,\ \beta_1 = |\angle ABD|,$ and $\beta_2 = |\angle CBD|$. By assumption (i),

$$\beta = \beta_1 + \beta_2.$$

By assumption (ii),

$$\delta = \delta_1 + \delta_2$$

where $\delta = |\angle ADC|,\ \delta_1 = |\angle ADB|$, and $\delta_2 = |\angle CDB|$. Finally, let $\alpha = |\angle DAB|$ and $\gamma = |\angle BCD|$. By Proposition 10, $\alpha + \beta_1 + \delta_1 = 180$ and $\gamma + \beta_2 + \delta_2 = 180$. So

$$360 = (\alpha + \beta_1 + \delta_1) + (\gamma + \beta_2 + \delta_2) = \alpha + (\beta_1 + \beta_2) + \gamma + (\delta_1 + \delta_2) = \alpha + \beta + \gamma + \delta.$$

□

[1]Since $D * A * E$, the points D and E are on opposite sides of $\overleftrightarrow{AC}$. Switching names if necessary, we can assume D and B are on the same side, and E and B are on opposite sides of $\overleftrightarrow{AC}$. In particular, ConAIA applies to $\angle EAC$ and $\angle BCA$.

Since m is parallel to $\overleftrightarrow{BC}$, the points B and C are on the same side of $\overleftrightarrow{AD}$. By definition of interior, we have that B is interior to $\angle DAC$. By a result of Incidence-Betweenness Geometry, C is interior to $\angle EAB$. Thus C and E are on the same side of $\overleftrightarrow{AB}$. So C and D are on opposite sides of $\overleftrightarrow{AB}$. In particular, ConAIA applies to $\angle DAB$ and $\angle CBA$.