IBC GEOMETRY

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This document develops the part of geometry that can be build on the axioms of incidence (I), betweenness (B), and congruence (C). We call this geometry *IBC Geometry*.

The axioms of IBC Geometry are a subset of Hilbert's axioms for Euclidean (and Hyperbolic) geometry. IBC Geometry does not include axioms for completeness or parallelism, but it includes everything else. I have made a few minor changes in Hilbert's original axioms, but the resulting geometry is equivalent.

This document builds on an earlier document about Incidence-Betweenness Geometry, and I assume that the reader is familiar with that document. In this edition, there are no drawings to illustrate the axioms, definitions, propositions, and proofs. I strongly recommend that you supply your own diagrams. The very act of producing the drawings will help your understanding. The only caveat is that in Hilbert style geometry, unlike Euclid style geometry, our proofs should not require facts that are obvious from the drawings; there should be no need to appeal to visual or geometric intuition.

1. The Axioms of IBC Geometry

IBC Geometry, as developed here, consists of 5 primitive terms, 14 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. All additional terms must be defined using primitive terms, previously defined terms, and basic set theory. We adopt all the notation and definitions from Incidence-Between Geometry including terms such as *line segment* or *angle*.

Axiom (Primitive Terms)**.** *The basic type of object is the point. Lines are sets of points. Betweenness is a three place relation of points. If P, Q, R are points, then $P * Q * R$ denotes the statement that the betweenness relation holds for (P, Q, R). Segment congruence is a two place relation of line segments, and angle congruence is a two place relation of angles. If $\overline{AB}$ and $\overline{CD}$ are line segments, then $\overline{AB} \cong \overline{CD}$ denotes the statement that the segment congruence relation holds between $\overline{AB}$ and $\overline{CD}$. If α and β are angles, then $\alpha \cong \beta$ denotes the statement that the angle congruence relation holds between α and β.*

Congruence between line segments and congruence between angles are primitive (undefined) relations. However, congruence between triangles is defined (as we will see below).

The axioms of IBC Geometry include the above axiom, I-1, I-2, I-3, B-1, B-2, B-3, B-4, and 6 new *congruence axioms* C-1, C-2, C-3, C-4, C-5, C-6. See the Betweenness-Incidence Geometry Handout for a statement of I-1 to I-3, and B-1 to B-4. Since the axioms of Betweenness-Incidence Geometry are a subset of the axioms of IBC Geometry, all the propositions of the previous document on Betweenness-Incidence Geometry are automatically propositions of IBC Geometry, and we will make heavy use of such propositions in what follows.

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Euclid's first common notion ("things which are equal to the same thing are also equal to each other"), is the inspiration for the transitive law for congruences. In fact, congruence should be reflexive and symmetric as well as transitive. This leads us to the first congruence axiom.

Axiom (C-1)**.** *Segment congruence is an equivalence relation for line segments.*

The next axiom concerns copying a given line segment onto a ray. It is inspired by Euclid's second postulate and his Proposition 3 (in Book I of the *Elements*).

Axiom (C-2)**.** *Suppose that* $\overline{AB}$ *is a line segment and* $\overrightarrow{CD}$ *is a ray. Then there is a unique point* E *on* $\overrightarrow{CD}$*, distinct from* C*, such that* $\overline{AB} \cong \overline{CE}$*.*

The next axiom concerns copying dividing or intermediate points on a segment.

Axiom (C-3)**.** *Suppose that* $\overline{AC}$ *and* $\overline{A'C'}$ *are congruent line segments. If* B *is a point such that* $A * B * C$*, then there is a point* B' *such that* $A' * B' * C'$*,* $\overline{AB} \cong \overline{A'B'}$*, and* $\overline{BC} \cong \overline{B'C'}$*.*

The next three axioms concern congruence of angles. The first of the three is similar to C-1 and is also inspired by Euclid's common notions. (Euclid's common notions were "common". They were supposed to apply to all types of "magnitudes" including segments/lengths and angles, but also numbers (integers) and volumes. The modern approach is not to assume that all magnitudes will automatically have the same properties, but instead to *prove* that they share properties.)

Axiom (C-4)**.** *Angle congruence is an equivalence relation for angles.*

The following axiom concerns copying a given angles onto a ray. It corresponds to Euclid's Proposition I-23.

Axiom (C-5)**.** *Suppose* $\angle BAC$ *is an angle, and* $\overrightarrow{DE}$ *is a ray. Then on any given side of* $\overleftrightarrow{DE}$*, there is a unique ray* $\overrightarrow{DF}$ *such that* $\angle BAC \cong \angle EDF$*.*

The final congruence axiom concerns copying of triangles. Before we give this axiom, we need a few definitions concerning triangles.

Definition 1. Given a triangle $\triangle ABC$, we sometimes denote $\angle BAC$ as just $\angle A$. We define $\angle B$ and $\angle C$ similarly.

Definition 2. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *congruent* if (and only if[1]) (i) $\overline{AB} \cong \overline{A'B'}$, (ii) $\overline{BC} \cong \overline{B'C'}$, (iii) $\overline{CA} \cong \overline{C'A'}$, (iv) $\angle A \cong \angle A'$, (v) $\angle B \cong \angle B'$, and (vi) $\angle C \cong \angle C'$.[2]

Even though *congruence* is a primitive term for line segments and angles, the above shows that it is a defined term for triangles.

In a few propositions (SAS I-5 and SSS I-8), Euclid assumes that triangles can be moved or copied. He did so without an explicit postulate. This idea is the inspiration for the following axiom which asserts that we can copy a given triangle onto a congruent base.

[1]Recall the convention that an if statement used to define a new term is really an if and only if statement.

[2]Technically, congruence is not a property of triangles themselves, but of triangles with a given ordering of their vertices. Triangles that are congruent under one ordering, might not be under other orderings. Triangles are obviously congruent to themselves when we use the same ordering. As we will see in the proof of Theorem 4, for isosceles triangles we have a triangle congruent to itself under a different order.

Axiom (C-6)**.** *Suppose $\triangle ABC$ is a triangle, and $\overline{A'B'}$ is a segment such that $\overline{AB} \cong \overline{A'B'}$. Then on any given side of $\overleftrightarrow{A'B'}$, there is a point C' such that $\triangle ABC \cong \triangle A'B'C'$.*

Remark. In Axiom C-6, I have chosen to depart from Hilbert's system. Hilbert choose SAS as the final congruence axiom. Inspired by Euclid, I decided to make SAS into a proposition with a proof inspired by that of Euclid's Proposition I-4. In order to make Euclid's proof valid according to our modern style of geometry, I had to add the above axiom concerning the copying of triangles.[3]

Axiom C-3 and C-6 do not assert uniqueness. This is because uniqueness can be proved.

Proposition 1. *The point B' in Axiom C-3 is unique.*

Proof. The point B' is on the ray $\overrightarrow{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$. By the uniqueness assertion of Axiom C-2, there can be no other such point. □

Proposition 2. *The point C' in Axiom C-6 is unique.*

Proof. Suppose C'' is another such point. By definition of triangle congruence, $\angle BAC \cong \angle B'A'C'$ and $\angle BAC \cong \angle B'A'C''$. By uniqueness in Axiom C-5, $\overrightarrow{A'C'} = \overrightarrow{A'C''}$. Also by definition of triangle congruence, $\overline{AC} \cong \overline{A'C'}$ and $\overline{AC} \cong \overline{A'C''}$. By the uniqueness assertion of axiom C-2, $C' = C''$. □

The following is essentially Proposition I-4 in the *Elements*.

Proposition 3 (SAS)**.** *Suppose $\triangle ABC$ and $\triangle A'B'C'$ are triangles such that (i) $\overline{AB} \cong \overline{A'B'}$, (ii) $\overline{AC} \cong \overline{A'C'}$, and (iii) $\angle A \cong \angle A'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. By Axiom C-6 there is a point D such that $\triangle ABC \cong \triangle A'B'D$. Furthermore, by Axiom C-6, this point can be chosen so that D and C' are on the same side of $\overleftrightarrow{A'B'}$.

By assumption $\angle BAC \cong \angle B'A'C'$, and by definition of triangle congruence $\angle BAC \cong \angle B'A'D$. By uniqueness in Axiom C-5, $\overrightarrow{A'C'} = \overrightarrow{A'D}$. By assumption $\overline{AC} \cong \overline{A'C'}$, and by definition of triangle congruence, $\overline{AC} \cong \overline{A'D}$. By uniqueness in Axiom C-2, $C' = D$.

Since $C' = D$ and $\triangle ABC \cong \triangle A'B'D$, we conclude that $\triangle ABC \cong \triangle A'B'C'$. □

As an application of SAS we prove the following famous theorem of geometry (Euclid's Prop. I-5, but with a much simpler proof due to the later Greek geometer Pappus).

Theorem 4 (Isosceles Base Angles)**.** *If $\triangle ABC$ is such that $\overline{AB} \cong \overline{AC}$, then $\angle B \cong \angle C$.*

Proof. By assumption (i) $\overline{AB} \cong \overline{AC}$. By Axiom C-1(symmetric) (ii) $\overline{AC} \cong \overline{AB}$. By Axiom C-4(reflexive) (iii) $\angle BAC \cong \angle CAB$. Thus $\triangle ABC \cong \triangle ACB$ by Proposition 3. In particular, $\angle B \cong \angle C$ (definition of triangle congruence). □

Definition 3. An *isosceles triangle* is one that has two sides that are congruent. The above shows that two of the angles must also be congruent. An *equilateral triangle* is one that has all three sides congruent.

From the above theorem we get the following.

[3]This is not my only departure from Hilbert. Many authors, myself included, have introduced various small modifications to Hilbert system. For example, Hilbert's version of C-3 is essentially Proposition 7 below. Ultimately, however, the geometry developed by these various modifications is essentially the same.

Corollary 5. *An equilateral triangle has all three angles congruent.*

Exercise 1. Prove the above.

Remark. We have not proved yet that isosceles and equilateral triangles exist. The above results tell us something about them if they do happen to exist. We cannot hope to prove the existence of equilateral triangles until later after we have established that certain circles intersect (as in Euclid's Prop. I-1). The intersection of circles requires an axiom beyond that of IBC geometry. The required axiom will be described in a future handout.

The term "congruence" is usually used only for equivalence relations. We have three types of congruences: segment, angle, and triangle. The first two are equivalence relations by assumption (Axioms C-1 and C-4). The last can be *proved* to be an equivalence relation.

Proposition 6. *Congruence* $\cong$ *is an equivalence relation among triangles (or more precisely, among triangles with a chosen ordering of the vertices).*

Exercise 2. Prove the above Proposition.

2. Basic Properties of Segments

Euclid's Common Notion 2 states that "if equals are added to equals, then the wholes are equal". The following proves that this is indeed the case for line segments.

Proposition 7 (Segment Addition)**.** *Suppose that* $A * B * C$ *and* $A' * B' * C'$*. If* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{BC} \cong \overline{B'C'}$ *then* $\overline{AC} \cong \overline{A'C'}$*.*

Proof. By Axiom C-2 there is a point E on $\overrightarrow{AC}$ such that $\overline{A'C'} \cong \overline{AE}$. By the symmetry law of Axiom C-1, we have $\overline{AE} \cong \overline{A'C'}$. If we can show that $E = C$, the result follows.

Since $A' * B' * C'$, Axiom C-3 gives us a point F on $\overline{AE}$ that matches B'. In other words, $A * F * E$, $\overline{A'B'} \cong \overline{AF}$ and $\overline{B'C'} \cong \overline{FE}$. By a result of incidence-betweenness geometry, $\overrightarrow{AC} = \overrightarrow{AE}$. Observe that this ray contains points B and F. Also $\overline{A'B'} \cong \overline{AF}$, and $\overline{A'B'} \cong \overline{AB}$ (by assumption and Axiom C-1). Thus $B = F$ by the uniqueness assertion of Axiom C-2.

Since $B = F$, we have $A * B * E$, and $\overline{B'C'} \cong \overline{BE}$. By results of incidence-betweenness geometry, $\overrightarrow{BE} = \overrightarrow{BC}$.[4] This ray contains points E and C, $\overline{B'C'} \cong \overline{BE}$, and $\overline{B'C'} \cong \overline{BC}$ (by assumption and Axiom C-1). Thus $E = C$ by the uniqueness assertion of Axiom C-2. Thus $\overline{AC} \cong \overline{A'C'}$. □

Remark. The above proof mentioned Axiom C-1 every time it was used. This practice has the effect of lengthening proofs without really adding new insight. From now on we will mostly use Axioms C-1 and C-4 without explicit mention. In other words, we will use the reflexive, symmetric, and transitive properties for $\cong$ as freely as we do for $=$.

Now we prove *segment subtraction* which is inspired by Euclid's Common Notion 3 ("if equals are subtracted from equals, then the remainders are equal").

Proposition 8 (Segment Subtraction)**.** *Suppose that* $A * B * C$ *and* $A' * B' * C'$*. If* $\overline{AC} \cong \overline{A'C'}$ *and* $\overline{AB} \cong \overline{A'B'}$ *then* $\overline{BC} \cong \overline{B'C'}$*.*

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[4] One way to see this is to use three and four point betweenness; there are three possible cases for A, B, C, E.

Proof. By Axiom C-3, there is a point X such that $A' * X * C'$, $\overline{AB} \cong \overline{A'X}$, and $\overline{BC} \cong \overline{XC'}$. Observe that $B', X \in \overrightarrow{A'C'}$. By the uniqueness assertion in Axiom C-2, $X = B'$. Since $\overline{BC} \cong \overline{XC'}$, the result follows. □

We end this section by defining an order $<$ among segments, and proving standard properties including transitivity and trichotomy. These were used as common notions by Euclid, although mostly not explicitly stated in his section called Common Notions.

Definition 4 ($<$ for Segments)**.** Let $\overline{AB}$ and $\overline{CD}$ be segments. We say that $\overline{AB} < \overline{CD}$ (or $\overline{CD} > \overline{AB}$) if there is a point E such that $C * E * D$ and $\overline{AB} \cong \overline{CE}$.

Proposition 9 (Substitution)**.** *If $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{C'D'}$ then $\overline{AB} < \overline{C'D'}$. Likewise, if $\overline{AB} < \overline{CD}$ and $\overline{AB} \cong \overline{A'B'}$ then $\overline{A'B'} < \overline{CD}$.*

Proof. Suppose that $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{C'D'}$. By Definition 4, there is a point E such that $C * E * D$ and $\overline{AB} \cong \overline{CE}$. By Axiom C-3, there is a point E' such that $C' * E' * D'$ and $\overline{CE} \cong \overline{C'E'}$. So $\overline{AB} \cong \overline{C'E'}$. Hence $\overline{AB} < \overline{C'D'}$ (Definition 4).

Suppose that $\overline{AB} < \overline{CD}$ and $\overline{AB} \cong \overline{A'B'}$. By Definition 4, there is a point E such that $C * E * D$ and $\overline{AB} \cong \overline{CE}$. So $\overline{A'B'} \cong \overline{CE}$. Thus $\overline{A'B'} < \overline{CD}$ (Definition 4). □

Euclid's Common Notion 5 states that 'the whole is greater than its parts.' The following shows that this is true for segments with our definition of $<$.

Proposition 10. *If $A * B * C$ then $\overline{AB} < \overline{AC}$.*

Proof. This follows directly from Definition 4 together with the fact that $\overline{AB} \cong \overline{AB}$. □

An important property of $<$ is transitivity:

Proposition 11 (Transitivity)**.** *If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$ then $\overline{AB} < \overline{EF}$.*

Proof. By definition of $<$, there is a point G such that $C * G * D$ and $\overline{AB} \cong \overline{CG}$. Likewise, there is a point H such that $E * H * F$ and $\overline{CD} \cong \overline{EH}$. By Axiom C-3, there is a point G' such that $E * G' * H$ and $\overline{EG'} \cong \overline{CG}$. So E-G'-H-F. In particular $E * G' * F$. Observe that $\overline{AB} \cong \overline{EG'}$ (since $\overline{EG'} \cong \overline{CG}$ and $\overline{AB} \cong \overline{CG}$). Thus $\overline{AB} < \overline{EF}$. □

We conclude this section with the Trichotomy property for segments. The proof of this property requires the following lemma.

Lemma 12. *If $\overline{AB}$ is a line segment, then $\overline{AB} < \overline{AB}$ does not hold.*

Proof. Suppose $\overline{AB} < \overline{AB}$. By definition of $<$, there is a point C such that $A * C * B$ and $\overline{AB} \cong \overline{AC}$. Observe that C and B are both on the ray $\overrightarrow{AB}$, and that $\overline{AB} \cong \overline{AC}$. By uniqueness in Axiom C-2, we have $B = C$. This contradicts $A * C * B$ and Axiom B-1. □

Proposition 13 (Trichotomy)**.** *Given segments $\overline{AB}$ and $\overline{CD}$ then exactly one of the following occurs: (i) $\overline{AB} < \overline{CD}$, (ii) $\overline{AB} \cong \overline{CD}$, (iii) $\overline{CD} < \overline{AB}$.*

Proof. By Axiom C-2 there is a unique point E on $\overrightarrow{CD}$ such that $\overline{AB} \cong \overline{CE}$. Since E is on $\overrightarrow{CD}$ we have either (a) $C * E * D$, (b) $E = D$, or (c) $C * D * E$. In the first case $\overline{AB} < \overline{CD}$ by definition of $<$. In the second case $\overline{AB} \cong \overline{CD}$. In the third case $\overline{CD} < \overline{CE}$ by Proposition 10. But $\overline{CE} \cong \overline{AB}$, so $\overline{CD} < \overline{AB}$ (Proposition 9).

We have shown at least one of (i), (ii), (iii) occur, now we must show at most one occurs. Suppose (i) and (iii) hold. Then $\overline{AB} < \overline{AB}$ by transitivity. Suppose (i) and (ii) hold. Then $\overline{AB} < \overline{AB}$ by Proposition 9. Suppose (ii) and (iii) hold. Then $\overline{AB} < \overline{AB}$ by Proposition 9. In any case $\overline{AB} < \overline{AB}$ holds, contradicting Lemma 12. □

3. Basic Properties of Angles

Recall how *supplementary angle* was defined (in the Inclusion-Betweenness Geometry handout): if $B * A * C$, and if D is a point not on $\overleftrightarrow{BC}$, then $\angle BAD$ and $\angle DAC$ are called *supplementary angles*.

The following is a basic result about supplementary angles. Its proof is an amusing argument using SAS three times.

Proposition 14. *Suppose α and β are supplementary angles, and that α' and β' are also supplementary angles. If $\alpha \cong \alpha'$ then $\beta \cong \beta'$.*

Proof. Write $\alpha = \angle BAD$ and $\beta = \angle DAC$ where $B * A * C$. Likewise write $\alpha' = \angle B'A'D'$ and $\beta' = \angle D'A'C'$ where $B' * A' * C'$. Let X be the point on $\overrightarrow{A'B'}$ such that $\overline{A'X} \cong \overline{AB}$ (Axiom C-2). Replace B' with X, so now $\overline{A'B'} \cong \overline{AB}$. Likewise, we can assume that $\overline{A'C'} \cong \overline{AC}$, and that $\overline{A'D'} \cong \overline{AD}$.

By SAS (Prop. 3), $\triangle BAD \cong \triangle B'A'D'$. Thus $\overline{BD} \cong \overline{B'D'}$ and $\angle B \cong \angle B'$.

By Segment Addition (Prop. 7), $\overline{BC} \cong \overline{B'C'}$. So, by SAS, $\triangle DBC \cong \triangle D'B'C'$. This implies that $\angle C \cong \angle C'$ and $\overline{CD} \cong \overline{C'D'}$. Since $\overline{AC} \cong \overline{A'C'}$, we can use SAS a third time to get that $\triangle ACD \cong \triangle A'C'D'$. This implies that $\beta \cong \beta'$. □

Definition 5 (Vertical Angles)**.** Suppose l and m are two lines intersecting at a point A. Suppose B and D are on m, and C and E are on l such that $B * A * D$ and $C * A * E$. Then $\angle BAC$ and $\angle DAE$ are called *vertical angles*.

Remark. The angles $\angle BAC$ and $\angle DAE$ in the above are both supplementary to the same angle, namely $\angle BAE$.

Remark. The definition of vertical angle doesn't use the concept of congruence. It could have been given in Incidence-Betweenness Geometry, but wasn't needed there.

The following is Proposition I-15 of the *Elements*. The proof is similar, but with important differences. For instance, we do not need to use angle addition or subtraction in any way.

Proposition 15 (Vertical Angles)**.** *Vertical angles are congruent.*

Exercise 3. Prove the above theorem appealing to Proposition 14.

Remark. In the following proposition, and in many other parts of this handout, it is important that you make diagrams. Of course, you cannot use the diagram to justify any steps of the proof, but it does help you understand what is going on.

The following proposition is related to Euclid's Proposition I-14.

Proposition 16. *Let $\overleftrightarrow{AD}$ be a line, and let B and C be on opposite sides of $\overleftrightarrow{AD}$. Suppose α and β are supplementary angles such that $\alpha \cong \angle BAD$ and $\beta \cong \angle DAC$. Then B, A, C are collinear. Thus $\angle BAD$ and $\angle DAC$ are also supplementary.*

Proof. Let E be a point such that $B * A * E$ (Axiom B-2). So B and E are on opposite side of $\overleftrightarrow{AD}$. By assumption B and C are on opposite sides of $\overleftrightarrow{AD}$. Thus C and E are on the same side of $\overleftrightarrow{AD}$.

Observe that $\angle BAD$ and $\angle DAE$ are supplementary. Since $\angle BAD \cong \alpha$ it follows that $\angle DAE \cong \beta$ by Proposition 14. By assumption $\angle DAC \cong \beta$. So by Axiom C-5, we have $\overrightarrow{AC} = \overrightarrow{AE}$. Thus C is on $\overleftrightarrow{AE} = \overleftrightarrow{AB}$. Hence A, B, C are collinear.

Since C and B are on opposite sides of $\overleftrightarrow{AD}$, there is a point $X \in \overleftrightarrow{AD}$ such that $B * X * C$ (definition of opposite sides). But B, C, A are collinear, and $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect in at most one point. Thus $X = A$. This implies $\angle BAD$ and $\angle DAC$ are also supplementary (definition of supplementary). □

Recall that Proposition 7 (Angle Addition) shows that segments satisfy Common Notion 2 in Book I of the *Elements* (‘if equals are added to equals, then the wholes are equal’). The following shows that angles also satisfy this property.

Proposition 17 (Angle Addition)**.** *Suppose that D is interior to $\angle BAC$, and that D' is interior to $\angle B'A'C'$. If $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$ then $\angle BAC \cong \angle B'A'C'$.*

Proof. By the Crossbar Theorem, there is a point E on the ray $\overrightarrow{AD}$ such that $B * E * C$. In particular B and C are on opposite sides of $\overleftrightarrow{AD}$. Likewise, B' and C' are on opposite sides of $\overleftrightarrow{A'D'}$.

By Axiom C-2 there is a point X on $\overrightarrow{A'B'}$ so that $\overline{A'X} \cong \overline{AB}$. By replacing B' with X, we can assume $\overline{A'B'} \cong \overline{AB}$. Likewise, replacing if necessary, we can assume that $\overline{A'C'} \cong \overline{AC}$. By Axiom C-2 again there is a point E' be on $\overrightarrow{A'D'}$ such that $\overline{A'E'} \cong \overline{AE}$. Note that even though B, E, C are collinear, we have not yet established that B', E', C' are collinear.

By SAS, $\triangle BAE \cong \triangle B'A'E'$ and $\triangle EAC \cong \triangle E'A'C'$. In particular $\angle BEA \cong \angle B'E'A'$ and $\angle AEC \cong \angle A'E'C'$. But $\angle BEA$ and $\angle AEC$ are supplementary. By Proposition 16, we get that B', E', C' are collinear. So $B' * E' * C'$ (since B' and C' are on opposite sides of $\overleftrightarrow{A'D'}$). Since $\triangle BAE \cong \triangle B'A'E'$ and $\triangle EAC \cong \triangle E'A'C'$, we get that $\angle ABE \cong \angle A'B'E'$ and $\overline{BE} \cong \overline{B'E'}$ and $\overline{EC} \cong \overline{E'C'}$. By Segment Addition (Prop. 7), $\overline{BC} \cong \overline{B'C'}$. Since $\overline{AB} \cong \overline{A'B'}$, we use SAS to conclude that $\triangle ABC \cong \triangle A'B'C'$. So $\angle BAC \cong \angle B'A'C'$. □

Next, we give the following which is the angle version of Axiom C-3.

Proposition 18. *Suppose that $\angle BAC \cong \angle B'A'C'$, and suppose that $\overrightarrow{AD}$ is a ray such that $\overrightarrow{AB} - -\overrightarrow{AD} - -\overrightarrow{AC}$. Then there is a ray $\overrightarrow{A'D'}$ such that $\overrightarrow{A'B'} - -\overrightarrow{A'D'} - -\overrightarrow{A'C'}$, $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$.*

Proof. Let B'' be a point on $\overrightarrow{A'B'}$ such that $\overline{AB} \cong \overline{A'B''}$ (Axiom C-2). Likewise, let C'' be a point on $\overrightarrow{A'C'}$ such that $\overline{AC} \cong \overline{A'C''}$ (Axiom C-2). By SAS (Theorem 3), $\triangle ABC \cong \triangle A'B''C''$. Thus $\overline{BC} \cong \overline{B''C'}$, $\angle B \cong \angle B''$, and $\angle C \cong \angle C''$.

By the Crossbar Theorem there is a point E on $\overrightarrow{AD}$ such that $B * E * C$. By Axiom C-3, there is a point E' such that $B'' * E' * C''$, $\overline{BE} \cong \overline{B''E'}$, and $\overline{EC} \cong \overline{E'C''}$. By the Crossbar-Betweenness Proposition, E' is in the interior of $\angle B''A'C'' = \angle B'A'C'$. Since $\overline{AB} \cong \overline{A'B''}$ and $\angle B \cong \angle B''$ and $\overline{BE} \cong \overline{B''E'}$, we have $\triangle ABE \cong \triangle A'B''E'$ (SAS). Thus $\angle BAE \cong \angle B''A'E'$. In other words, $\angle BAD \cong \angle B'A'E'$. Likewise, $\angle DAC \cong \angle E'A'C'$.

Let $D' = E'$. Since D' is in the interior of $\angle B'A'C'$ we have $\overrightarrow{A'B'} - -\overrightarrow{A'D'} - -\overrightarrow{A'C'}$. Also, with this choice of D', we have $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$. □

Exercise 4. Show that the ray $\overrightarrow{A'D'}$ in the above proposition is the unique ray with the required property. Hint: the proof of the following might give a clue.

Proposition 19 (Angle Subtraction)**.** *Suppose that D is interior to $\angle BAC$, and that D' is interior to $\angle B'A'C'$. If $\angle BAD \cong \angle B'A'D'$ and $\angle BAC \cong \angle B'A'C'$ then $\angle DAC \cong \angle D'A'C'$.*

Proof. By Prop. 18, there is a ray $\overrightarrow{A'D''}$ where $\overrightarrow{A'B'} - -\overrightarrow{A'D''} - -\overrightarrow{A'C'}$ and $\angle BAD \cong \angle B'A'D''$ and $\angle DAC \cong \angle D''A'C'$. We will prove the result by showing that $\overrightarrow{A'D'} = \overrightarrow{A'D''}$.

By definition of betweenness of rays, D'' is in the interior of $\angle B'A'C'$. By definition of interior of an angle this implies that $D'' \sim_l C'$ where $l = \overleftrightarrow{A'B'}$. Similarly $D' \sim_l C'$. Thus $D'' \sim_l D'$. By the uniqueness statement of Axiom C-5, $\overrightarrow{A'D'} = \overrightarrow{A'D''}$. □

4. Ordering Angles by Size

Now we define and investigate inequality for angles. We will follow the pattern of inequalities for segments in Section 2.

Definition 6 ($<$ for Angles)**.** Let $\angle BAC$ and $\angle EDF$ be angles. We say $\angle BAC < \angle EDF$ (or $\angle EDF > \angle BAC$) if there is a ray $\overrightarrow{DG}$ such that $\overrightarrow{DE}$ -- $\overrightarrow{DG}$ -- $\overrightarrow{DF}$ and $\angle BAC \cong \angle EDG$. In other words, there is a point G is in the interior of $\angle EDF$ such that $\angle BAC \cong \angle EDG$.

Proposition 20 (Substitution)**.** *Let $\alpha, \alpha', \beta, \beta'$ be angles. If $\alpha < \beta$ and $\alpha \cong \alpha'$ then $\alpha' < \beta$. If $\alpha < \beta$ and $\beta \cong \beta'$ then $\alpha < \beta'$.*

Proof. The first claim is simple. Suppose $\beta = \angle CBD$. Then there is a point X interior to $\angle CBD$ so that $\alpha \cong \angle CBX$. Thus $\alpha' \cong \angle CBX$ (Axiom C-4). Thus $\alpha' < \angle CBD$.

The second claim requires use of Proposition 18. Suppose $\beta = \angle CBD$ and $\beta' = \angle C'B'D'$. By definition of $<$ there is a point X interior to $\angle CBD$ so that $\alpha \cong \angle CBX$. By Proposition 18 there is a point X' in the interior of $\angle C'B'D'$ so that $\angle C'B'X' \cong \angle CBX$. Thus $\alpha \cong \angle C'B'X'$ (Axiom C-4). So, by definition, $\alpha < \angle C'B'D'$. □

Euclid's Common Notion 5 states that 'the whole is greater than its parts.' The following shows that this is true for angles with our definition of $<$ (and thinking of angle interiors when saying the word 'part').

Proposition 21. *If $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ then $\angle BAD < \angle BAC$.*

Proof. This follows directly from Definition 6 together with the fact that $\angle BAD \cong \angle BAD$. □

An important property of $<$ is transitivity:

Proposition 22 (Transitivity)**.** *If α, β, γ are angles such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

Proof. Write $\beta = \angle CBD$ and $\gamma = \angle FEG$. By definition of $<$, there is a ray $\overrightarrow{BX}$ such that $\overrightarrow{BC}$ -- $\overrightarrow{BX}$ -- $\overrightarrow{BD}$ and $\alpha \cong \angle CBX$, and there is a ray $\overrightarrow{EY}$ such that $\overrightarrow{EF}$ -- $\overrightarrow{EY}$ -- $\overrightarrow{EG}$ and $\beta \cong \angle FEY$. By Proposition 18, there is a ray $\overrightarrow{EX'}$ such that $\overrightarrow{EF}$ -- $\overrightarrow{EX'}$ -- $\overrightarrow{EY}$ and $\angle FEX' \cong \angle CBX$. Observe that $\angle FEX' \cong \alpha$.

By the squeezing property of ray betweenness (of Incident-Betweenness Geometry),

$$\overrightarrow{EF} \text{ -- } \overrightarrow{EX'} \text{ -- } \overrightarrow{EG}.$$

Thus $\alpha < \angle FEG$. □

Before proving trichotomy, it is useful to have the following:

Proposition 23. *Suppose that α and β are supplementary angles, and that α' and β' are also supplementary angles. If $\alpha > \alpha'$ then $\beta < \beta'$.*

Proof. Write $\alpha = \angle BAD$ and $\beta = \angle DAC$ where $B * A * C$. Since $\alpha' < \alpha$, there is a ray $\overrightarrow{AE}$ such that $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD}$ and $\angle BAE \cong \alpha'$. By Proposition 14, $\angle EAC \cong \beta'$.

Since $B * A * C$ and $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD}$, we have $\overrightarrow{AE}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ (by a proposition from Incidence-Betweenness Geometry). By Proposition 21, $\angle DAC < \angle EAC$. But $\angle DAC = \beta$, so $\beta < \angle EAC$. Since $\angle EAC \cong \beta'$, it follows that $\beta < \beta'$ (Proposition 20). □

Lemma 24. *Given an angle α, it cannot happen that $\alpha < \alpha$.*

Proof. Suppose $\alpha < \alpha$. Write $\alpha = \angle BAD$. By definition of $<$ there is a point X in the interior of $\angle BAD$ such that $\angle BAX \cong \angle BAD$. By definition of angle interior, X and D must be on the same side of $\overleftrightarrow{AB}$. By the uniqueness claim of Axiom C-5, $\overrightarrow{AX} = \overrightarrow{AD}$ contradicting that X is interior to $\angle BAD$. □

Proposition 25 (Angle Trichotomy)**.** *Given angles α and β, exactly one of the following occurs: (i) $\alpha < \beta$, (ii) $\alpha \cong \beta$, (iii) $\beta < \alpha$.*

Proof. Let $B * A * C$ be points on a line (Axioms I-2, B-2). Let D and E be points on the same side of $\overleftrightarrow{BC}$ such that $\angle BAD \cong \alpha$ and $\angle BAE \cong \beta$ (Axiom C-4). By the Supplementary Interior Proposition (Part 2) of Inclusion-Betweenness Geometry, One of the following must occur: (i) E is in the interior of $\angle BAD$, (ii) E is on the ray $\overrightarrow{AD}$, (iii) E is in the interior of the supplementary angle $\angle DAC$.

In case (i), $\angle BAE < \angle BAD$ (Proposition 21). So $\alpha < \beta$ (Proposition 20). In case (ii), $\angle BAE = \angle BAD$. So $\alpha \cong \beta$ (transitivity of $\cong$). Finally, in case (iii), $\angle EAC < \angle DAC$ (Proposition 21). So by Proposition 23, $\angle BAD < \angle BAE$. So $\beta < \alpha$ (Proposition 20).

We have shown at least one of (i), (ii), (iii) occur. Now we must show at most one occurs. Suppose (i) and (iii) hold. Then $\alpha < \alpha$ by transitivity. Suppose (i) and (ii) hold. Then $\alpha < \alpha$ by substitution. Suppose (ii) and (iii) hold. Then $\alpha < \alpha$. We conclude that, in any of these cases, $\alpha < \alpha$ holds. This contradicts Lemma 24. □

5. Right Angles and Perpendicular Lines

The following is similar to a definition in the *Elements* (Definition 10 of Book I).

Definition 7. Let α be an angle. Suppose α has a supplementary angle congruent to itself, then α is said to be a *right angle*.

Here is a basic consequence of the definition of right angle.

Proposition 26. *Let α be a right angle. (i) If $\gamma \cong \alpha$ then γ is right. (ii) If γ is supplementary to α then γ is right.*

Proof. By Definition 7, α has a supplementary angle β such that $\alpha \cong \beta$.

To prove the first claim, suppose $\gamma \cong \alpha$. Let δ be supplementary to γ (Supplementary Existence Proposition in Incidence-Betweenness Geometry). By Proposition 14, $\delta \cong \beta$. By transitivity of congruences (Axiom C-4), $\gamma \cong \delta$. Thus γ is right.

To prove the second claim, observe that β might not be equal to γ (or else it would be easy). We need to use Proposition 14: since $\alpha \cong \alpha$ it follows that $\beta \cong \gamma$. Since $\alpha \cong \beta$, $\gamma \cong \alpha$. So γ is congruent to its supplementary angle. Thus γ is right. □

Euclid took as a postulate, his Postulate 4, that all right triangles are congruent. We do not need to take this as an axiom, but we can make it into a theorem.

Proposition 27. *Any two right angles are congruent.*

Proof. Let α be a right angle with supplementary angle β such that $\alpha \cong \beta$. Let α' be a right angle with supplementary angle β' such that $\alpha' \cong \beta'$. Our goal is to show $\alpha \cong \alpha'$.

Suppose instead that $\alpha < \alpha'$. Then $\beta' < \beta$ by Proposition 23. Thus $\beta' < \alpha$ by subsititution using $\alpha \cong \beta$ (Proposition 20), and again $\alpha' < \alpha$ by substitution using $\alpha' \cong \beta'$. Since $\alpha < \alpha'$ and $\alpha' < \alpha$, we contradict angle trichotomy (Proposition 25).

We have shown that $\alpha < \alpha'$ cannot hold. Similarly, $\alpha' < \alpha$ cannot hold. Thus, by angle trichotomy (Proposition 25) we must have that $\alpha \cong \alpha'$. □

Definition 8 (Perpendicular Line)**.** Two lines l and m are said to be *perpendicular* if the following holds: the lines intersect at a point A, and there is a point $B \neq A$ on l and a point $C \neq A$ on m such that $\angle BAC$ is a right angle. If l and m are perpendicular, we sometimes write $l \perp m$.

Remark. Suppose l and m are perpendicular because one choice of B and C resulted in a right angle $\angle BAC$. Then any other choice of B and C will also give a right angle. To see this, suppose that B is replaced by B' for example. If B' is on the ray $\overrightarrow{AB}$ then we get the same angle, but if B' is on the opposite ray then we get supplementary angles. However, we proved above that supplementary angles of right angles are also right (Proposition 26).

Now we consider the existence of perpendicular lines and right angles. The following is related the Euclid's Proposition I-12.

Proposition 28. *If l is a line and P is a point not on l then there is a line m passing through P that is perpendicular to l.*

Remark. The above can be extended to give existence *and* uniqueness. This will be done in the next section.

Proof. Let A and B be two points on l (Axiom I-2). let $\overrightarrow{AP'}$ be a ray on the side of l not containing P such that $\angle BAP \cong \angle BAP'$ (Axiom C-5). By Axiom C-2, we can choose P' along this ray in such a way that $\overline{AP} \cong \overline{AP'}$. Since P and P' are on opposite sides of l, there is a point $Q \in l$ such that $P * Q * P'$.

If $A = Q$, then $\angle BQP \cong \angle BQP'$. Since $\angle BQP$ and $\angle BQP'$ are supplementary, they are, by definition, right angles. So l is perpendicular to $m = \overleftrightarrow{PQ}$ and we are done. So from now on assume that $A \neq Q$.

Observe that $\angle BAP = \angle QAP$ if B is on $\overrightarrow{AQ}$, otherwise $\angle BAP$ and $\angle QAP$ are supplementary. A similar observation holds for $\angle BAP'$ and $\angle QAP'$. Since $\angle BAP \cong \angle BAP'$ it follows (from Proposition 14 if necessary) that $\angle QAP \cong \angle QAP'$. So, by SAS, $\triangle AQP \cong \triangle AQP'$. In particular, $\angle AQP \cong \angle AQP'$. But these angles are supplementary since $P * Q * P'$. Thus $\angle AQP$ is right. By definition, l is perpendicular to $m = \overleftrightarrow{PQ}$. □

Corollary 29. *Right angles exist. Perpendicular lines exist.*

Proof. From incidence-betweenness geometry we know that there exists a line, and a point not on the line. Now apply the above theorem to this line and point. □

The following is related the Euclid's Proposition I-11.

Proposition 30. *If l is a line and P is a point on l then there is a unique line m passing through P that is perpendicular to l.*

Proof. Let α be a right angle. It exists by the previous corollary. Let Q be a point on l not equal to P (Axiom I-2). By Axiom C-5, there is a unique ray $\overrightarrow{PC}$ such that $\angle QPC \cong \alpha$. By Proposition 26, $\angle QPC$ is a right angle. So $m = \overleftrightarrow{PC}$ is perpendicular to l, and we have established the existence claim.

Suppose m' is perpendicular to l and contains P. Use Axiom I-2 and B-2 to find a point $C' \in l'$ so that C and C' are on the same side of l. Apply the uniqueness claims of Axiom C-5 to conclude that $m = m'$. The details are left to the reader. □

Exercise 5. Fill in the details in the proof of the uniqueness claim for the above proposition.

6. Parallel Lines

Recall from the previous handout that distinct lines are parallel if and only if they do not intersect. Parallel lines are an important concept in this course. In this section, we prove a few basic results that do not depend on any particular parallel axiom. We begin with the Alternate Interior Angle Theorem, which is essentially Euclid's Proposition I-27.

Theorem 31 (Alternate Interior Angle)**.** *Suppose C and D are on opposite sides of $\overleftrightarrow{AB}$. If $\angle CAB \cong \angle DBA$, then $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel.*

Proof. Suppose $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are not parallel. Our first goal is to show that if they intersect on one side of the line $\overleftrightarrow{AB}$ they have to intersect on both sides.

So suppose that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at a point X on the side of $\overleftrightarrow{AB}$ containing C. So X is on the ray $\overrightarrow{AC}$. Observe that $D * B * X$ since D and X are on opposite sides of $\overleftrightarrow{AB}$. By Axiom C-2 there is a point Y on $\overrightarrow{BD}$ such that $\overline{BY} \cong \overline{AX}$.

By SAS, $\triangle ABX \cong \triangle BAY$. In particular $\angle ABX \cong \angle BAY$. Now since X and Y are on opposite sides of l, and X, Y, B are collinear, we have $X * B * Y$. So the angle $\angle ABX$ is supplementary to $\angle ABY$. By Proposition 16, and the fact that $\angle YBA \cong \angle XAB$ and $\angle ABX \cong \angle BAY$, we must have that X, A, Y are collinear. Thus Y is on $\overleftrightarrow{AC} = \overleftrightarrow{AX}$.

We have established that both X and Y are on both $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. The points X and Y are distinct since they are on opposite sides of $\overleftrightarrow{AB}$. By Axiom I-1 (uniqueness) we have $\overleftrightarrow{AC} = \overleftrightarrow{XY} = \overleftrightarrow{BD}$. This line intersects $\overleftrightarrow{AB}$ in both A and B, contradicting the earlier result that distinct lines intersect in at most one point. From this contradiction we conclude that $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$ □

Corollary 32. *If l and l' are distinct lines perpendicular to the same line m. Then $l \parallel l'$.*

Proof. Let A be the intersection of l with m. Let B be the intersection of l' with m. By the uniqueness claim of Proposition 30, $A \neq B$. Let C be a point on l not equal to A (Axiom I-2). Let D be a point on l' not equal to B (Axiom I-2). We can assume that D is on the side of l not containing C (use Axiom B-2 if necessary to replace D with another point).

Since $l \perp m$, we have $\angle CAB$ is right. Since $l' \perp m$ we have $\angle DBA$ is right. By Proposition 27, $\angle CAB \cong \angle DBA$. By the Alternate Interior Angle Theorem, $l \parallel l'$. □

Proposition 28 can now be strengthened to include existence.

Proposition 33 (Existence and Uniqueness of Perpendiculars)**.** *If l is a line and P is a point not on l then there is a unique line m passing through P that is perpendicular to l.*

Proof. Existence was proved in Proposition 28. So now suppose m and m' are distinct lines, perpendicular to l and containing P. By the above corollary, they are parallel. This contradicts the fact that they both contain P. □

Finally, we can show that parallel lines exist. This is related to Euclid's Proposition I-31.

Proposition 34 (Existence of Parallels)**.** *Given a line l and a point P not on l, there is at least one line l' that is parallel to l.*

Exercise 6. Prove the above theorem. Hint: use Proposition 33 followed by Proposition 30 and Corollary 32.

7. Inequalities Involving Triangles

Now we study a variety of results all related to comparisons of sides and angles of triangles. We begin with the Exterior Angle Theorem, which is Euclid's Proposition I-16.

Theorem 35 (Exterior Angle)**.** *Given a triangle $\triangle ABC$ and a point D such that $B * C * D$, then the exterior angle $\angle ACD$ is greater than both opposite interior angles $\angle A$ and $\angle B$.*

Proof. We divide this into two theorems: part (i) $\angle ACD > \angle A$, and part (ii) $\angle ACD > \angle B$.

To prove part (i) we suppose it fails. By trichotomy, this means either $\angle ACD \cong \angle A$ or $\angle ACD < \angle A$. If $\angle ACD \cong \angle A$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ by the Alternate Interior Angle Theorem. However, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at the point B, a contradiction.

Suppose that $\angle ACD < \angle A$. Then, by definition there is a ray $\overrightarrow{AE}$ interior to $\angle A$ such that $\angle CAE \cong \angle ACD$. By the Crossbar Theorem, we can choose E so that $B * E * C$. Note that D and E are on opposite sides of $\overleftrightarrow{AC}$ (why?) and that $\angle ACD \cong \angle CAE$. So by the Alternate Interior Angle Theorem $\overleftrightarrow{CD} \parallel \overleftrightarrow{AE}$. However, E is on both $\overleftrightarrow{CD}$ (why?) and $\overleftrightarrow{AE}$, a contradiction.

In either case, we get a contradiction. Thus $\angle ACD > \angle A$, finishing the proof of part (i).

Now we consider part (ii) of the theorem. Let F be a point such that $A*C*F$. Then $\angle ACD$ and $\angle BCF$ are vertical angles. So $\angle ACD \cong \angle BCF$ by the Vertical Angles Proposition (Proposition 15). Now apply part (i) of the current theorem to the case of $\angle BCF$ opposite to $\angle B$. So $\angle BCF > \angle B$. By substitution (Proposition 20), $\angle ACD > \angle B$. □

Exercise 7. Show that D and E, in part (i) of the above proof, are on opposite sides of $\overleftrightarrow{AC}$ as claimed.

One corollary of this theorem is that the other two angles of a right triangle are acute. First some definitions.

Definition 9. If $\triangle ABC$ is a triangle with right angle $\angle A$, then $\triangle ABC$ is called a *right triangle*, the sides $\overline{AB}$ and $\overline{AC}$ are called *legs*, and $\overline{BC}$ is called the *hypotenuse*.

Definition 10. Let β be a right angle. If α is an angle such that $\alpha < \beta$ then α is an *acute angle*. If α is an angle such that $\alpha > \beta$ then α is an *obtuse angle*.

Proposition 36. *If $\triangle ABC$ is a right triangle with right angle $\angle A$, then $\angle B$ and $\angle C$ are both acute angles.*

Exercise 8. Use the Exterior Angle Theorem to prove the above theorem.

Euclid's Proposition I-18 asserts that given two angles of a triangle, the larger is the angle opposite the larger side. Proposition I-19 asserts that given two sides, the larger is the side opposite the larger angle. These are combined here in the following:

Proposition 37. *Let $\triangle ABC$ be a triangle. Then $\overline{BC} > \overline{AC}$ if and only if $\angle A > \angle B$.*

Proof. Divide this into two propositions: part (i) $\overline{BC} > \overline{AC} \implies \angle A > \angle B$, and part (ii) the converse.

Part (i). If $\overline{BC} > \overline{AC}$ then by definition of $<$ there is a point D such that $C * D * B$ and $\overline{CD} \cong \overline{AC}$. By the Crossbar-Betweenness Theorem, D is interior to $\angle A$, so $\angle CAD < \angle A$ (Proposition 21). By the Isosceles Base Angles Theorem (Theorem 4), $\angle CAD \cong \angle ADC$. By the Exterior Angle Theorem (Theorem 35), $\angle ADC > \angle B$. By the transitive and substituation properties of $<$, we get $\angle A > \angle B$.

Part (ii). Suppose $\angle A > \angle B$. By trichotomy, $\overline{BC} \cong \overline{AC}$ or $\overline{BC} < \overline{AC}$ or $\overline{AC} < \overline{BC}$. If $\overline{BC} \cong \overline{AC}$ then $\angle A \cong \angle B$ by the Isosceles Base Angles Theorem (Theorem 4), which contradicts the hypothesis that $\angle A > \angle B$. If $\overline{BC} < \overline{AC}$, then apply part (i) of the current proposition to conclude that $\angle A < \angle B$. This contradicts the hypothesis that $\angle A > \angle B$. We get contradictions in the other cases, so $\overline{BC} > \overline{AC}$. □

The following exercise is useful in the proof of the Proposition 39 below:

Exercise 9. Suppose $\overrightarrow{AD} = \overrightarrow{AE}$. In other words, suppose that D and E are non-vertex points on the same ray. Then $\overline{AD} < \overline{AE}$ implies $A * D * E$. Hint: use the definition of $<$ together with the uniqueness claim of Axiom C-2. Note: the converse statement was proved earlier in this handout.

The following lemma is useful in the proof of the proposition that follows.

Lemma 38. *Let $\triangle ABC$ be a triangle, and D a point such that $B * D * C$. If $\overline{AC} > \overline{AB}$ or $\overline{AC} \cong \overline{AB}$ then $\overline{AC} > \overline{AD}$.*

Proof. By the Exterior Angle Theorem (Theorem 35), $\angle ADC > \angle B$. If $\overline{AC} > \overline{AB}$ then $\angle B > \angle C$ by Proposition 37, so $\angle ADC > \angle C$ by transitivity. If $\overline{AC} \cong \overline{AB}$ then $\angle B \cong \angle C$ by Theorem 4, so $\angle ADC > \angle C$ by substitution. In either case $\angle ADC > \angle C$. Thus $\overline{AC} > \overline{AD}$ by Proposition 37. □

Exercise 10. Draw sketches illustrating the above lemma, and the following proposition.

Proposition 39. *Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangle such that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. Then $\angle BAC > \angle B'A'C'$ if and only if $\overline{BC} > \overline{B'C'}$.*

Remark. The above proposition combines Euclid's Propositions I-24 and I-25.

Proof. Part (i): from the assumption $\angle BAC > \angle B'A'C'$ we wish to show $\overline{BC} > \overline{B'C'}$.

By trichotomy, either $\overline{AC} \cong \overline{AB}$ or one of $\overline{AC}$ and $\overline{AB}$ is greater than the other; without loss of generality assume $\overline{AC} \cong \overline{AB}$ or $\overline{AC} > \overline{AB}$. By the definition of $>$ there is a point D interior to $\angle A$ such that $\angle BAD \cong \angle B'A'C'$, and by the Crossbar Theorem we can choose D such that $B * D * C$. By Lemma 38, $\overline{AC} > \overline{AD}$.

Let E be a point of the ray $\overrightarrow{AD}$ such that $\overline{AE} \cong \overline{AC}$ (Axiom C-2). By substitution, $\overline{AE} > \overline{AD}$, so $A * D * E$ (Exercise 9). This implies that B, C, E are non-collinear, so we can legitimately speak of the triangle $\triangle BCE$.

Since $B * D * C$, the Crossbar-Betweenness Proposition implies that D is interior to $\angle CEB$, so $\angle CED < \angle CEB$ (Proposition 21). By the Isosceles Base Angle Theorem, $\angle AEC \cong \angle ACE$, but $\angle AEC$ is just $\angle CED$. So $\angle CED \cong \angle ACE$. By substitution into $\angle CED < \angle CEB$ we get $\angle ACE < \angle CEB$.

Since $A*D*E$, the Crossbar-Betweenness Proposition implies that D is interior to $\angle ACE$, so $\angle DCE < \angle ACE$ (Proposition 21). By transitivity, $\angle DCE < \angle CEB$. Now $\angle DCE$ is just $\angle BCE$, so $\angle BCE < \angle CEB$. By Proposition 37, $\overline{BE} < \overline{BC}$.

By SAS, $\triangle ABE \cong \triangle A'B'C'$. So $\overline{BE} \cong \overline{B'C'}$. By substitution, $\overline{B'C'} < \overline{BC}$ as desired.

Part (ii): from $\overline{BC} > \overline{B'C'}$ we wish to show $\angle A > \angle A'$. Suppose otherwise. By trichotomy of angles, this implies $\angle A \cong \angle A'$ or $\angle A < \angle A'$. Suppose $\angle A \cong \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$ by SAS. So $\overline{BC} \cong \overline{B'C'}$, a contradiction to trichotomy for segments. Finally, suppose $\angle A < \angle A'$. Then by part (i) of the current theorem, $\overline{BC} < \overline{B'C'}$, a contradiction to trichotomy for segments. In any case, we get a contradiction. So $\angle A > \angle A'$. □

Informally speaking, the following proposition says that the sum of two sides of a triangle is always greater than the third, and so corresponds to Euclid's Proposition I-20. Formally, we have not yet discussed the meaning of $\overline{XY} + \overline{ZW}$ for general segments. In the special case where $A * B * D$ you can think of $\overline{AB} + \overline{BD}$ as being $\overline{AD}$, but we will wait until a later document to officially define $+$ in the context of general segments. Observe that the symbol $+$ is nowhere formally defined in this document.

Proposition 40 (Triangle Inequality: first form)**.** *Let $\triangle ABC$ be a triangle, and let D be a point such that $A * B * D$ and $\overline{BD} \cong \overline{BC}$. Then $\overline{AC} < \overline{AD}$.*

Proof. Observe that $\triangle BCD$ is isosceles. So $\angle BCD \cong \angle BDC$.

Since $A*B*D$, the point B is interior to $\angle ACD$ by the Crossbar-Betweenness Proposition. So $\angle BCD < \angle ACD$ by Proposition 21. By substitution, $\angle BDC < \angle ACD$. Observe that $\angle BDC$ is just $\angle ADC$. So $\angle ADC < \angle ACD$. By Proposition 37, $\overline{AC} < \overline{AD}$. □

8. Triangle Congruence Theorems

We begin with Angle-Angle-Side which corresponds to the second part of Euclid's Proposition I-26.

Proposition 41 (AAS)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose $\overline{BC} \cong \overline{B'C'}$. Suppose also that $\angle B \cong \angle B'$ and $\angle A \cong \angle A'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. By Axiom C-6 there is a point D such that $\triangle DBC \cong \triangle A'B'C'$, and where D and A are on the same side of $\overleftrightarrow{BC}$. In particular, $\angle DBC \cong \angle A'B'C' \cong \angle ABC$. By the uniqueness assertion of Axiom C-5, $\overrightarrow{BD} \cong \overrightarrow{BA}$. So $B * A * D$ or $A = D$ or $B * D * A$.

If $B*A*D$ then $\angle BAC > \angle BDC$ by the Exterior Angle Theorem. But $\angle BDC \cong \angle A'$, so $\angle BAC > \angle A'$ by substitution. This contradicts our hypothesis (that $\angle A \cong \angle A'$). Likewise, $B * D * A$ leads to a contradiction.

Since $B * A * D$ and $B * D * A$ lead to contradiction, $A = D$. So $\triangle ABC \cong \triangle A'B'C'$. □

The following corresponds to the first part of Euclid's Proposition I-26.

Proposition 42 (ASA)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose $\overline{BC} \cong \overline{B'C'}$. Suppose also that $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. By Axiom C-6 there is a point D such that $\triangle DBC \cong \triangle A'B'C'$, and where D and A are on the same side of $\overleftrightarrow{BC}$. In particular, $\angle DBC \cong \angle A'B'C' \cong \angle ABC$. By the uniqueness assertion of Axiom C-5, $\overrightarrow{BD} \cong \overrightarrow{BA}$. A similar argument shows $\overrightarrow{CD} \cong \overrightarrow{CA}$.

So we have two lines: $\overleftrightarrow{BD} = \overleftrightarrow{BA}$ and $\overleftrightarrow{CD} = \overleftrightarrow{CA}$ that intersect in both D and A. The lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CA}$ can intersect in at most one point by an earlier result (they are distinct lines since A, B, and C are not collinear). So $A = D$. Since $\triangle DBC \cong \triangle A'B'C'$, we get $\triangle ABC \cong \triangle A'B'C'$ as desired. □

The following corresponds to Euclid's Proposition I-6.

Corollary 43. *If two distinct angles of a triangle are congruent then the triangle is isosceles. If all three angles of a triangle are congruent then the triangle is equilateral.*

Proof. Suppose $\triangle ABC$ is such that $\angle B \cong \angle C$. We have all the hypotheses needed for ASA: $\angle B \cong \angle C$ and $\overline{BC} \cong \overline{CB}$ and $\angle C \cong \angle B$. Thus, by ASA, $\triangle ABC \cong \triangle ACB$. So $\overline{AB} \cong \overline{AC}$ and the triangle is isosceles.

If in addition $\angle A \cong \angle B$, then a similar argument gives $\overline{BC} \cong \overline{AC}$. So all three sides are congruent and the triangle is equilateral. □

The following corresponds to Euclid's Proposition I-8.

Proposition 44 (SSS)**.** *If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ and $\overline{CA} \cong \overline{C'A'}$, where $\triangle ABC$ and $\triangle A'B'C'$ are triangles, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. Since $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$, we can apply Proposition 39. Suppose $\angle A > \angle A'$, then Proposition 39 gives us that $\overline{BC} > \overline{B'C'}$, contradiction our hypothesis. Similarly the supposition $\angle A' > \angle A$ gives a contradiction. Thus, by trichotomy, we must have $\angle A \cong \angle A'$. By SAS, $\triangle ABC \cong \triangle A'B'C'$. □

Exercise 11. Draw a counter-example to angle-side-side. Do so by drawing triangles $\triangle ABC$ and $\triangle ABC'$ that differ in only one vertex, and make B, C, and C' collinear. You do not have to rigorously justify your counter-example, but make it clear from the drawing that it is a counter-example.

Although there is no ASS theorem in geometry (at least not without an extra hypothesis), for right triangles we do get such a theorem.

Proposition 45 (HL)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be right triangles with right angles $\angle A$ and $\angle A'$. If $\overline{BC} \cong \overline{B'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. By Axiom C-6 there is a point D such that $\triangle ABD \cong \triangle A'B'C'$, and we can choose D to be on the side of $\overleftrightarrow{AB}$ not containing C. Observe that $\triangle ABD$ is a right triangle.

Let α and β be supplementary right angles. These exist by Corollary 29. Since all right triangles are congruent (Proposition 27) we have $\angle BAC \cong \alpha$ and $\angle BAD \cong \beta$. By Proposition 16, to C, A, D are collinear.

The line through C, A, D cannot contain B since A, B, C are non-collinear. Thus C, D, B cannot be collinear, and $\triangle CDB$ exists. Since $\overline{BC} \cong \overline{B'C'}$ and $\overline{B'C'} \cong \overline{BD}$, the triangle $\triangle CDB$ is isosceles. So $\angle DCB \cong \angle CDB$ (Theorem 4).

Observe that $\angle CDB = \angle ADB$ and $\angle DCB = \angle ACB$ (since $C * A * D$). We conclude that $\angle ACB \cong \angle ADB$. Since $\angle CAB$ and $\angle DAB$ are right, they are congruent (Proposition 27). So by AAS we have $\triangle ABC \cong \triangle ABD$. But $\triangle ABD \cong \triangle A'B'C'$, so $\triangle ABC \cong \triangle A'B'C'$. □

9. Midpoints and Bisectors

Definition 11 (Midpoint)**.** Let $\overline{AB}$ be a line segment. A *midpoint* of $\overline{AB}$ is a point M such that $A * M * B$ and $\overline{AM} \cong \overline{MB}$.

We will now prove that midpoints exist. First we give three lemmas.

Lemma 46. *If a segment has a midpoint, then it is unique.*

Proof. Suppose M and M' are distinct midpoints of $\overline{AB}$. Without loss of generality, we assume A-M-M'-B. So $M * M' * B$, which implies $\overline{M'B} < \overline{MB}$ (Proposition 10). Likewise, $A * M * M'$, which implies $\overline{AM} < \overline{AM'}$. Since $\overline{AM'} \cong \overline{M'B}$, $\overline{AM} < \overline{M'B}$ by substitution. By transitivity, we get $\overline{AM} < \overline{MB}$, contradicting the definition of midpoint. □

The next two lemmas concern the construction used in the main proposition.

Lemma 47. *Suppose that $\overline{AB}$ is a segment and that C and D are points on opposite sides of $\overleftrightarrow{AB}$ such that $\angle BAC$ and $\angle ABD$ are right. Then $\overline{CD}$ cannot contain A or B.*

Proof. If $A \in \overline{CD}$ then A, C, D are collinear. So D is on the line $\overleftrightarrow{AC}$. By the Alternate Interior Angle Theorem, $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$, but these lines intersect in D. This is a contradiction. Likewise $B \in \overline{CD}$ leads to a contradiction. □

Lemma 48. *Suppose that $\overline{AB}$ is a segment and that C and D are points on opposite sides of $\overleftrightarrow{AB}$ such that $\angle BAC$ and $\angle ABD$ are right. Then $\overline{CD}$ must intersect $\overleftrightarrow{AB}$ is a point M such that $A * M * B$.*

Proof. Since C and D are on opposite sides of $\overleftrightarrow{AB}$, the segment $\overline{CD}$ must intersect $\overleftrightarrow{AB}$ at some point M. From Lemma 47, A, B, M are distinct. By Axiom B-3, either $M * A * B$ or $A * M * B$ or $A * B * M$.

Suppose $M * A * B$. Then $\triangle MAC$ is a right triangle with right angle $\angle MAC$ (Proposition 26). By Proposition 36, $\angle AMC$ is acute. However, $\angle AMC$ is an external angle to $\triangle MBD$, and by the External Angle Theorem $\angle AMC > \angle MBD$. However, $\angle MBD = \angle ABD$ is a right angle. Thus $\angle AMC$ is both acute and obtuse, a contradiction.

A similar contradiction occurs if $A * B * M$. Thus $A * M * B$ is the only possibility. □

Proposition 49 (Midpoint Existence)**.** *Every segment has a unique midpoint.*

Proof. Let $\overline{AB}$ be a segment. Our goal is to show that $\overline{AB}$ has a unique midpoint. Let $\overleftrightarrow{AC}$ be a line perpendicular to $\overleftrightarrow{AB}$ containing the point A. Such a line exists by Proposition 30. By the definition of perpendicular line (and the remark that follows it), we have that $\angle BAC$ is a right angle. By Axiom C-5, there is a ray $\overrightarrow{BD}$ such that $\angle ABD \cong \angle BAC$, and such that C and D are on opposite sides of $\overleftrightarrow{AB}$. By Proposition 26, $\angle ABD$ is right. By Axiom C-2, we can choose D so that $\overline{AC} \cong \overline{BD}$.

By Lemma 48, the segment $\overline{CD}$ intersects $\overleftrightarrow{AB}$ at a point M with $A * M * B$. Observe that $C*M*D$, so by the Vertical Angles Proposition, $\angle AMC \cong \angle BMD$. Also $\angle MAC \cong \angle MBD$ since they are both right, and $\overline{AC} \cong \overline{BD}$. Thus by AAS, $\triangle AMC \cong \triangle BMD$. In particular, $\overline{AM} \cong \overline{BM}$. We conclude that M is a midpoint of $\overline{AB}$. It is the unique midpoint by Lemma 46. □

Remark. The above proposition corresponds to Euclid's Proposition I-10. However, our proof is different: it doesn't require the existence of equilateral triangles.

Definition 12 (Angle Bisector)**.** Let $\angle BAC$ be an angle. A *bisector* of $\angle BAC$ is a ray $\overrightarrow{AD}$ such that $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ and $\angle BAD \cong \angle DAC$.

The following proposition corresponds to Euclid's Proposition I-9.

Proposition 50 (Bisector Existence)**.** *Every angle has a unique bisector.*

Proof. Let $\angle BAC$ be an angle. Let C' be a point on $\overrightarrow{AC}$ such that $\overline{AX} \cong \overline{AB}$ (Axiom C-2). We replace C with X if necessary, and so assume that $\triangle BAC$ is isosceles with $\overline{AC} \cong \overline{AB}$. Let D be the midpoint of $\overline{BC}$ (Proposition 49). We leave it to the reader to show that $\overrightarrow{AD}$ is a bisector.

Now we show uniqueness. Suppose that $\overrightarrow{AD'}$ is also a bisector. Then $\overrightarrow{AD'}$ intersects $\overline{BC}$ by the Crossbar Theorem. By replacing D' with this intersection point, we can assume that $B * D' * C$. We leave it to the reader to show that D' is a midpoint of $\overline{BC}$. But D is also a midpoint. So $D = D'$ by the uniqueness claim of Proposition 49. This shows that the bisector is unique. □

Exercise 12. Show that $\overrightarrow{AD}$ in the above proof is a bisector.

Exercise 13. Show that D' in the above proof is a midpoint of $\overline{BC}$.

Definition 13 (Perpendicular Bisector)**.** Let $\overline{AB}$ be a segment. A *perpendicular bisector* to $\overline{AB}$ is a line l that (i) is perpendicular to $\overleftrightarrow{AB}$ and, (ii) intersects $\overleftrightarrow{AB}$ in the midpoint of $\overline{AB}$.

Proposition 51. *Every segment $\overline{AB}$ has a unique perpendicular bisector.*

Exercise 14. Prove the above proposition.

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