THERE IS NO LARGEST PRIME NUMBER

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In earlier classes we proved the following handy facts:

**Lemma.** Let $a, b, c$ be integers. If $a|b$ and $a|c$, then $a$ divides any linear combination of $b$ and $c$. In other words, $a|ub + vc$ for all $u, v \in \mathbb{Z}$.

**Lemma.** If $a, b$ are positive integers and if $a|b$, then $a \leq b$.

We will need these facts to prove the following important theorem (from Sept. 14):

**Theorem.** There is no largest prime number.

**Remark.** Since every non-empty finite set of integers has a maximum, this result shows that the set of primes cannot be finite. In other words, there are an infinite number of primes.

**Proof.** (By contradiction) Suppose that $P$ is the largest prime number. Let $N = P! + 1$. By the Fundamental Theorem of Arithmetic, $N$ is the product of primes. Let $q$ be a prime that occurs in the prime factorization of $N$. So $q$ is a divisor of $N = P! + 1$.

Since $q$ is a prime number, $q \leq P$ because $P$ is the largest prime. Now $P! = 1 \cdot 2 \cdot 3 \cdots P$, so every positive integer less than or equal to $P$ is a divisor of $P!$. In particular $q$ divides $P!$.

Now we are in a very strange situation: $q|P! + 1$ and $q|P!$. This implies that $q$ divides the linear combination $1 \cdot (P! + 1) + (-1) \cdot P! = 1$. Thus $q|1$. This means $q \leq 1$ which is a contradiction since primes are greater than 1.

Here is a concise version of the proof:

**Proof.** Suppose $P$ is the largest prime number, and let $N = P! + 1$. Let $q$ be a prime divisor of $N$. Observe that $q | P!$ since $q \leq P$. Thus $q$ divides $N - P! = 1$. Contradiction.

One strategy for learning proofs is to learn concise versions, and test yourself to see if you can justify each step.

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