ESSAY ON CONSTRUCTING NUMBERS

PROFESSOR WAYNE AITKEN

This is an essay based on class notes for Math 330. It is still a relatively early draft. For this draft, many of the results I quote are drawn from memory, but I plan to add citations to more authoritative sources in future drafts.

1. The Greek Notion of Number

In discussing the construction of numbers, I will use real numbers for convenience, but it is important to realize that this is a deviation from Greek mathematical practice. In this section I discuss the Greek practice just to set the record straight. In future sections, I will generally ignore the Greek practice and adopt a more modern style for convenience.

The ancient Greeks around the time of Euclid did not have our notion of real or rational numbers. Also, they did not use negative numbers, complex numbers, or 0. In fact, the only type of number that they considered were positive integers:

$$1, 2, 3, 4, \ldots$$

In particular, the Greek word $\alpha\rho\iota\theta\mu o\varsigma$ (*arithmos*, 'number') only referred to positive integers. In some sources even 1 (unity), and sometimes 2 as well, were not classified as numbers per se: they were understood and used, but considered too simple to be numbers.

In spite of the absence of other number systems, the Greeks were able to develop a very impressive body of mathematics. To get by without rational and real numbers, they used the related notions of *magnitudes*, *ratios*, and *proportions* to carry out their mathematics.

Warning. When I say "Greeks", I really mean theoretical mathematicians around the time of Euclid. There is evidence (whose merits are debated by scholars) that later Greek used fractions as we do. Also mathematicians involved in applied fields including astronomy might have come closer to our notion of a real number.

1.1. **Magnitudes.** The Greeks did not use real numbers, instead they used magnitudes. They had five basic types of magnitudes: integers, lengths, areas, volumes, and angles. The common notions of Euclid were intended to apply equally to all of them. Today we view all five types of magnitudes as being real numbers (when units are chosen), or at least measurable by

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real numbers. The Greeks did not think of them as all unified under one number system, but thought of them as five distinct types of magnitudes. The Greeks would never say that a length L was greater than an area A since size only makes sense among magnitudes of the same type. On the other hand, we can say such things today, at least theoretically, since we view L and A as being real numbers.

1.2. **Ratios and Proportions.** Instead of using rational numbers, the ancient Greeks used ratios of integers which were written $n : m$. They did not just consider ratios of integers, but they considered also considered ratios of magnitudes of the various types (integers, lengths, areas, volumes, angles). However, to the Greeks, *the ratio* $x : y$ *is defined only if* x *and* y *are magnitudes of the same type.* Thus, you could not have x an area and y a length. This would seem absurd to an ancient mathematician. In modern English we use the expression "comparing apples and oranges" for this sort of nonsensical comparison.

You can compare ratios. For example,

$$a : b \ :: \ c : d$$

means that $a : b$ and $c : d$ are equivalent ratios. Another way of saying this is that a and b are in the same *proportion* as c and d. The idea of proportion arose early on in the study of similar triangles where it is the key idea. The notation $a : b \ :: \ c : d$ is old fashioned so I will write $a : b = c : d$ instead for the rest of this section.

Ratios are very close to our notion of fraction, and so whenever I want to modernize a statement made by Greek mathematicians I often convert ratios to fractions. For example, if n and m are integers, then the ratio $n : m$ can be thought of as representing the fractions $\frac{n}{m}$. Keep in mind, though that the Greeks did not think of ratios as fractions. In fact, they did not think of ratios as numbers at all, but as a comparison between numbers or magnitudes.

Many of our laws of fractions were known to the Greeks in terms of ratios. If a, b, c, d are positive integers, then we have the law that

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc$$

but the Greeks would instead use the law

$$a : b = c : d \qquad \text{if and only if} \qquad ad = bc.$$

This law works best with positive integers. For lengths (segments) they would have to be more careful, since a length times a length was thought of as a rectangular area, not another length. The law for segments is the statement that

$$a : b = c : d$$

if and only if the a by d rectangle is equal (in area) to the b by c rectangle.

To take things farther, if L and L' are segments and R and R' are rectangles, then the rules is that $L : L' = R : R'$ if and only if the box with base R and height L' is equal (in volume) to the box with base R' and height L. This shows that different types of objects can be compared. So the expression $a : b = c : d$ does not presuppose that a, b, c, d are all four of the same type, but only that a and b are of the same type, and that c and d are of the same type.

As another example, the Greeks were aware of the law that

$$a : b = c : d \qquad \text{implies} \qquad a : c = b : d$$

but only if a, b, c, d are of the same type. This corresponds to our law of fractions

$$\frac{a}{b} = \frac{c}{d} \qquad \text{implies} \qquad \frac{a}{c} = \frac{b}{d}.$$

The Greeks were able to do mathematics without fractions and rational numbers because much of what you do with fractions can be done with ratios. However, you should not think that ratios and fractions are the exact same thing. For example, it would never have occurred to a Greek mathematician to add ratios, but we add fractions all the time.

The conventional wisdom is that the early Pythagoreans believed at first that regardless of the type of object, the ratio $x : y$ was equal to a ratio $m : n$ of integers. In other words, they believed that all magnitudes of the same type were commensurable. We say two objects (of the same type) are *commensurable* if their ratio is equal to a ratio of integers. The Pythagoreans later discovered incommensurable lengths, for example the the diagonal of a square and its side. So ratios of lengths or other types of magnitudes do not reduce to the study of ratios of integers. Ancient mathematicians such as Eudoxus had to develop a more complicated theory of ratio and proportion (equality between ratios) to allow for incommensurable magnitudes. For example, the Pythagorean proof of the Pythagorean theorem might have made use of ratios via similar triangles, but it was later not considered rigorous until the theory of proportion was developed by Eudoxus. In fact, Euclid, wanted to prove the Pythagorean theorem (Book I) before he developed proportions (Book V), and so had to find a different proof that did not appeal to similar triangles.

Incommensurability is closely related to our notion of *irrational.* In fact, two magitudes are incommensurable if and only if the quotient of their real number measures is an irrational number.

There are lots of examples of incommensurable lengths and many were known by Euclid's time. However, one of the most basic examples, the fact that the the circumference of a circle is incommensurable with its diameter, was not proved in ancient times. In fact, it wasn't proved until 1761 by the mathematician Lambert. Today we express this fact by saying that π is irrational.

2. Constructible Numbers

In what follows, I will go against Greek practice and use real numbers. This makes it easier for us to understand the nature of the Greek constructions, even if it is anachronistic.

In order to use the real numbers to modernize Greek mathematics, we need to fix a segment u called the unit. Actually, the Greeks did sometimes fix a unit. There is a theorem that once a unit is chosen then every length (line segment) can be assigned a real number in such a way that (i) u is assigned 1, and (ii) a few other reasonable laws are satisfied (which I will skip for now, but I will cover in my Geometry course). We can likewise assign a real number to any rectangle representing areas in such a way that (i) a u by u square is assigned 1, and (ii) a few other reasonable laws, which I will skip, are satisfied. Not only can straight lines and rectangles be assigned real numbers in this way, but many curves (such as circles, or segments of parabolas) and regions (such as the area inside a circle) as well. Volumes and angles can be assigned real numbers as well.

So from now on fix a segment u and use real numbers (in a modern way) for other segments.

Definition 1. We say that a real number is *constructructible* if it is the length of a segment that can be constructed from u by means of a straightedge and compass.

Definition 2. The set of constructible numbers K is defined to be the set of all real numbers that are constructible.

Many numbers are constructible, including $\sqrt{n}$ for all integers n. For example, to construct $\sqrt{2}$ make a square of sides all equal to u and draw the diagonal. It is interesting that many notable real numbers are not constructible. For example, the cube root of 2 as well as the numbers π and e are not constructible. The nonconstructibility of the cube root of 2 is related to the Delian problem (duplicating the cube), and the nonconstructibility of π is related to the problem of squaring the circle.

If you randomly choose points you will have a segment, and it may happen to have length equal to the cube root of 2, or some other desired real number. However, this does not count as a constructible number. For a length to be considered constructible, you must have a foolproof method of constructing a segment of the desired length, and this method should involve straightedge and compass (in other words, lines and circles) used in the correct manner.

We can talk about constructing real numbers with other curves besides lines and circles (straightedge and compass), but these are constructible in a more general sense and might not be in K. To be in K you must be able to construct them only using lines and circles (not conics, spirals, etc.).

We will now explore various closure properties of constructible numbers.

3. Closure Properties

We begin with a theorem known to the ancient Greeks, and probably discovered some time before Euclid. It is a basic application of similar triangles.

Theorem 1. *Suppose a, b, c are given lengths. Then we can construct a segment d such that $b : a = d : c$.*

Proof. Using a and b, construct a right triangle $\triangle PQR$ of base equal to b and height a. Assume that $\overline{QP}$ is the height and $\overline{QR}$ is the base. Thus Q is the vertex of the right angle. On the ray $\overrightarrow{QP}$ draw a point S such that $\overline{QS}$ is equal to c. Draw a line parallel to the hypotenuse $\overline{PR}$ passing through S. This will intersect the ray $\overrightarrow{QR}$ at some point T. We can prove that $\triangle PQR$ is similar to $\triangle SQT$ using properties of parallel lines. Thus

$$\overline{QR} : \overline{QP} = \overline{QT} : \overline{QS}$$

In other words $b : a = d : c$ where d is $\overline{QT}$.

Note: all these constructions described in this proof can be executed with a straightedge and compass. □

This leads to several closure theorems which we will now consider.

Warning. The theorem above was known to Euclid, but the theorems below are not Euclidean. This is not due to the fact that they are hard, but rather because they require a more modern way of thinking, perhaps originating with Descartes in the 1600s. When Euclid multiplied segments he thought of the answer as a rectangular area, and he never divided segments by other segments. (He divided segments by integers, but not by other segments. Likewise, when he multiplied a segment by an integer, the result could be thought of as a segment.)

Theorem 2. *Suppose you are given segments of length b and c. Then you can construct a segment of length bc.*

Corollary 3. *The set K is closed under multiplication. In other words, if two real numbers are in K, then their product will also be in K.*

Proof. Given b and c, let $a = u$ be the unit segment. Then Theorem 1 states that we can construct d such that $b : u = d : c$. Translated into modern notation, this means that $b/1 = d/c$. Thus $d = bc$ as desired. □

Theorem 4. *Suppose you are given segments of length a and b. Then you can construct a segment of length b/a.*

Corollary 5. *The set K is closed under division. In other words, if two real numbers are in K, then their quotient will also be in K.*

Proof. Given a and b, let $c = u$ be the unit segment. Then the above theorem states that we can construct d such that $b : a = d : u$. Thus $b/a = d/1$. So $d = b/a$ as desired. □

We can construct segments of any positive integer length by laying segments of length u next to each other an arbitrary number of times on a common line. This gives the following.

Theorem 6. *Any positive integer is constructible. Thus K contains the set of positive integers.*

This theorem together with the fact that K is closed under division, shows that any positive rational number is constructible. Thus

Theorem 7. *Let $\mathbb{Q}_+$ be the set of positive rational numbers. Then $\mathbb{Q}_+ \subset K$.*

Later we will see that this inclusion is strict: there are numbers in K that are irrational. Before we do this, we prove some easy theorems.

Theorem 8. *Suppose you are given segments of length b and a. Then you can construct a segment of length $b + a$.*

Corollary 9. *The set K is closed under addition. In other words, if two real numbers are in K, then their sum will also be in K.*

Proof. Given a and b, draw a line and two segments of length a and b respectively on this line laid side to side. Then the combined segment has length $a + b$. □

Theorem 10. *Suppose you are given segments of length b and a. If $b > a$ then you can construct a segment of length $b - a$.*

Corollary 11. *The set K is partially closed under subtraction in the sense that given two distinct real numbers, the larger minus the smaller is in K.*

Proof. Let $\overline{PQ}$ be a segment of length b. Let $\overline{PR}$ be a subsegment of length a. Then $\overline{RQ}$ has length $b - a$ as desired. □

Note. (for readers who have had field theory). If a non-empty subset of $\mathbb{R}$ is closed under multiplication, division (by non-zero quotients), addition, and subtraction, then it is a subfield. Observe that K is not a subfield since it is only partially closed under subtraction: it does not contain the additive inverse of its elements. If we allow for zero and negative lengths, we can increase K to form a subfield E of $\mathbb{R}$ such that K consists of the positive elements of E. This field, which is sometimes called the *Euclidean field*, is interesting from the point of view of field theory. Using field theory we can show that E cannot contain the cube root of 2, which implies that the Delian problem cannot be solved with only straightedge and compass. Similar considerations show that the circle cannot be squared, and arbitrary angles cannot be trisected with only straightedge and compass. In fact, one can show that a sixty degree angle cannot be trisected, and that a twenty degree angle is not constructible. The study of constructible angles is interesting. It is related to Gauss' construction of the 17-gon, and more generally to Mersenne primes.

We give one more closure property.

Theorem 12. *Suppose you are given a segment of length a. Then you can construct a segment of length $\sqrt{a}$*

Corollary 13. *The set K is closed under square roots. In other words, if a real number is in K, then its square root is in K.*

Proof. Let m be the geometric mean of the unit u and a. Recall from our previous unit that this is constructible using a circle of diameter $a + u$ and a perpendicular to the diameter drawn through a certain point. Therefore, $a : m = m : u$ by definition of geometric mean. In modern notation, this is $a/m = m/1$. Thus $a = m^2$, so m is the square root. □

This shows that K contains some irrational numbers. For example, $\sqrt{2}$. Thus K is larger than $\mathbb{Q}_+$. However, with field theory one can show that the cube root of 2 is not in K. Thus K is not closed under cube roots.

Exercise 1. Show that K is closed under fourth roots. Hint: your answer should be really short.

4. Circles

The quadrature of the circle (squaring the circle) is a famous problems from antiquity. Given a circle with center O and radius $r = \overline{OP}$ it asks you to construct a square of equal area.

There is another interesting ancient problem that is less famous: to rectify the circle. By *rectify* I mean "straighten". Given a circle with center O and radius $r = \overline{OP}$ this problem asks you to construct a line segment of the same length as the circumference of the circle.

These two problems are closely related. In fact, *if you can solve either problem for one given circle, you can solve both for all circles.* By "solve" I do not mean necessarily with straightedge and compass, but with whatever methods and curves are admitted in the discussion. To be precise, then, the result should be restated as follows: if you can solve either problem for one given circle using some method, then you can use that solution together with a straightedge and compass to solve both problems for all circles.

We begin with rectification:

Theorem 14. *If you can solve the rectification problem for one particular circle, then you can solve it for any other circle. The solution for other circles requires the solution for the particular circle and, from that point, only the straightedge and compass.*

Proof. Suppose you are given a particular circle with radius r_0. Suppose also that you manage to rectify this one circle. Then you have a segment of length $2\pi r_0$. By closure under division (Theorem 4) you can now construct the quotient $2\pi r_0 / r_0 = 2\pi$ with straightedge and compass.

Suppose now that you are given another circle with radius r. Then you can construct the product $(2\pi)r$ by closure under multiplication (Theorem 2). □

Remark 1. Observe that we did not explicitly give a construction, but just appealed to closure under division and multiplication.

Exercise 2. Show that if you can rectify a circle, then you can construct π and if you can construct π you can rectify any circle. Hint: use closure properties.

Now we turn to the quadrature problem. First we review (without proof) a theorem found in the writings of Archimedes, but probably proved earlier, concerning the area of a circle.

Theorem 15. *Suppose you have a circle of radius r and circumference c. Suppose also that you have a triangle of base c and height r. Then the circle and triangle have the same area.*

Exercise 3. Derive the formula $A = \pi r^2$ for the area of the circle from the above theorem. Feel free to use formulas such as $c = 2\pi r$.

This might seem like a hopeful path to squaring the circle since if you can construct a triangle, you can construct a square of the same area (see the appendix). The catch is that you might not be able to construct the triangle! The first proof of the following shows that under certain circumstance you can construct the triangle.

Theorem 16. *If you can rectify one circle, then you can square any circle. The solution to the quadrature problem requires only the solution to the rectification of a circle and, from that point, only the straightedge and compass.*

Proof. Suppose you have a circle of radius r_0 that has been rectified. In other words you have a segment of length $2\pi r_0$. By closure under division (Theorem 4) you can construct the quotient $2\pi r_0/r_0 = 2\pi$.

Now let C be a circle of radius r. It is arbitrary: it could be the same circle that you rectified or any other given circle. Since you have r and you have 2π, you can construct the product $2\pi r$ by closure under multiplication (Theorem 2).

Now that you have constructed r and $2\pi r$, you can construct a right triangle with base $2\pi r$ and height r (it doesn't really have to be a right triangle, but these are easy to construct using perpendiculars). This triangle has the same area as the circle by Theorem 15. Now square this triangle (see Appendix). This square will square the circle. □

Second proof. Suppose you have a circle of radius r_0 that has been rectified. In other words you have a segment of length $2\pi r_0$. By closure under division (Theorem 4) you can construct the quotient $2\pi r_0/r_0 = 2\pi$.

You can construct 2 since it is an integer (Theorem 6). Thus you can construct $2\pi/2 = \pi$ by closure under division (Theorem 4). Now we can construct $\sqrt{\pi}$ using closure under square roots (Theorem 12).

Now let C be a circle of radius r. Since you have r and you have $\sqrt{\pi}$, you can construct the product $\sqrt{\pi}r$ by closure under multiplication (Theorem 2).

Finally use this to construct a square with sides equal to $\sqrt{\pi}r$. This square has area πr^2 which is the area of the circle C. □

Exercise 4. Show that if you can square one circle, then you can rectify any circle. Hint: look at the second proof above, but change the direction of your reasoning.

Exercise 5. Show that if you can square one circle, then you can square any circle.

5. The Spiral of Archimedes

Today we know that π is not in the set K; in other words, it is not constructible using straightedge and compass. This implies that we can neither square nor rectify a circle with only these tools. Dinostratus showed that if you use a new tool, the curve of Hippias (called the Quadratrix since it is related to quadrature) then you can rectify and square the circle.

I will skip Dinostratus's contribution and jump to Archimedes who did something similar with another curve called the *spiral of Archimedes*. This spiral is defined to be the curve traced out by a point the moves at a uniform angular velocity about a fixed pole O but at the same time increases its distance from O uniformly. Think of the second hand of a big clock, and think of an ant that starts at the center of the clock and walks slowly away along the second hand at a uniform speed. The path the ant traces out along his journey is the spiral of Archimedes. However, it is customary to proceed counter-clockwise: we want a clock that goes counter-clockwise!

We can use calculus and polar coordinates, two techniques not available in Archimedes day, to help illustrate his result in a convenient modern fashion. In polar coordinates his spiral is $r = k\theta$ where k is a positive constant and $\theta \geq 0$. This spiral was used to solve two problems (i) trisection of angle, and (ii) squaring a circle. We know today that neither can be solved with only a compass and straightedge

Theorem 17. *You can trisect any given angle using the spiral of Archimedes together with a compass and straightedge.*

Proof. The basic idea is that the spiral provides a linear relationship between angles and lengths. You can trisect segments with ruler and compass, and the spiral converts this trisection into an angle trisections. Not only can you trisect angles this way, but you can n-sect them for any integer $n \geq 2$.

Here are the details. Given an angle, use a straightedge and compass to draw a congruent angle with (i) its vertex at the pole of the spiral, (ii) one of its rays as the positive x-axis, and (iii) the second ray consisting of points with positive y coordinates. (I will skip the details of constructing congruent angles, but it is not hard to do).

Suppose this angle has measure θ_0. Then one of the rays has equation $\theta = 0$ and the other $\theta = \theta_0$. This second ray intersects the spiral at a point P with $r = k\theta_0$. In other words $\overline{OP}$ has length $k\theta_0$.

By Theorem 6 we can construct a segment of length 3. So, by closure under division (Theorem 4) we can construct a segment of length equal to $k\theta_0/3$.

Since we can construct a segment of length $k\theta_0/3$, we can draw a circle of radius $k\theta_0/3$ with center O. Where does this intersect the spiral? Obviously at a point Q where $r = k\theta_0/3$. Since $r = k\theta$ on the spiral, we have that $k\theta_0/3 = r = k\theta$. Thus Q has $\theta = \theta_0/3$. Thus the angle formed from the x-axis and the ray $\overrightarrow{OQ}$ has angle $\theta_0/3$ as desired. □

To square the circle, we don't want the spiral of Archimedes, but a tangent line to the spiral.

Exercise 6. Find the slope of the tangent lines to $r = k\theta$ when $\theta = \pi/2$ and when $\theta = 2\pi$. Hint: find a calculus book that discusses polar coordinates.

Exercise 7. Find the y-intercept Q of the tangent lines to $r = k\theta$ when $\theta = 2\pi$. Find the dimensions of the right triangle $\triangle POQ$ where P is the point on the spiral with $\theta = 2\pi$.

Exercise 8. Consider the triangle $\triangle POQ$ of the previous exercise, and the circle of center O and radius $\overline{OP}$. Show that the circle and the triangle have the same area. Show also that $\overline{OQ}$ equals the circumference of the circle.

This exercise shows that the spiral rectifies a particular circle. It also produces a triangle of the same area. Since we can square a triangle (see Appendix) we can square the circle.

By what we did earlier we know that this implies that we can square and rectify any circle from this particular rectification or squaring. This is essentially how Archimedes solved the problem.

As the above exercise showed, we can rectify a given circle. Archimedes proved something more general.

Theorem 18. *Let P be a point on the spiral, and C be the circle with center O and radius $\overline{OP}$. Let l be the tangent line to the spiral at P, and let m be the perpendicular to $\overleftrightarrow{OP}$ containing O. Let Q be the intersection of l and m. Then $\overline{OQ}$ rectifies the arc on the circle C that starts at the positive x-axis and ends at P.*

6. Appendix: Squaring rectangles, triangles, and polygons

6.1. **Triangles.** Squaring a triangle is not hard. Call one side the base b. Drop a perpendicular from the opposite vertex to b. This gives the height h. Now bisect either the base or the height, lets say we bisect h giving us

h'. Finally, use an earlier construction to find the geometric mean m of b and h' (or of b' and h if you bisect b).

By definition $b : m = m : h'$. In modern language, $b/m = m/h'$. Thus $m^2 = bh' = bh/2$. So the square with side m has the same areas as the triangle.

6.2. **Rectangles and parallelograms.** Squaring a rectangle is even easier. Call one side the base b and the other the height h. Let m be the geometric mean. By definition $b : m = m : h$. Thus $b/m = m/h$, so $m^2 = bh$. So the square with side m has the same areas as the rectangle.

A similar process works for parallelograms, except you must obtain the height h by dropping a perpendicular. Even trapezoids can be squared in this way: drop a perpendicular to find the height h, and find the arithmetic mean a of the two bases. Now find the geometric mean of h and a.

For more general quadrilaterals, use the procedure for polygons.

6.3. **Polygons.** To square a polygon, first divide the polygon into triangles $T_1, \dots, T_n$. Pick a base b_i for T_i, and drop a perpendicular to find the height h_i. Using multiplicative closure, we can construct $b_i h_i$. Using additive closure, we can construct $b_1 h_1 + \dots + b_n h_n$. Bisecting this gives a segment whose length x is equal (as real numbers) to the area of the polygon. Using closure by square roots we can construct $\sqrt{x}$. Now construct a square whose side is $\sqrt{x}$. It has area x, so it has area equal to the polygon.

This proof is quick and slick since it uses the closure properties. However, it relies on real numbers and it might not translate well into the Greek methodology which does not use real numbers. Euclid gives a different procedure via a method called *application of areas*. He shows how to convert each triangle T_i into a rectangle. Then he shows how to convert any collection of rectangles into rectangles all with a common side length (but of differing heights in general) using "application of areas". Finally he stacks these rectangles together into a big rectangle (which you can do since they have the same base length). Finally he converts this rectangle into a square.

7. Appendix: Other notes

This section does not really belong in this essay, but I am including since it contains information concerning the most recent homework assignment.

Let S be a sphere of radius R. Let C be a cylinder of base radius R and height $2R$. Then S fits in C. This is what appeared on Archimedes' tomb. Archimedes showed that the ratio of the volume of S to the volume of C is $2 : 3$. He also showed that the ratio of the surface area of S to the surface area of C is also $2 : 3$.

However, if you remove the top and bottom of C then the result is even simpler, the surface area of S is equal to the surface area of the side of the cylinder C.

This same result holds for sections of the sphere. Intersect the sphere with a plane, and let X be the part of the sphere on one side of the plane. Assume that the plane is horizontal to make things easier, and that X is above the plane. Let x be the distance from the top of the sphere to the circle that forms the edge of X. Here is what Archimedes proved:

Theorem 19. *The surface area of X is equal to the area of a circle of radius x.*

Our textbook said that this obviously implies the following:

Corollary 20. *Let h be the height of X. Then X has surface area equal to the side of the cylinder of base radius R and height h.*

This corollary shows that the result about the sphere and the cylinder generalizes to sections of the sphere and sections of the cylinder.

Exercise 9. Show that the corollary follows from the theorem. (You do not need to prove the theorem, but just derive the theorem from the corollary).