DAVENPORT PAIRS OVER FINITE FIELDS

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Abstract. A pair of polynomials $f, g \in \mathbb{F}_q[T]$ is called a *Davenport pair* (DP) if their value sets are equal, $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$, for *infinitely* many extensions of $\mathbb{F}_q$. If they are equal for *all* extensions of $\mathbb{F}_q$, i.e., for all $t \geq 1$, then we say that (f, g) is a *strong Davenport pair* (SDP). One may consider exceptional polynomials and SDP's as special cases of DP's. Exceptional polynomials and SDP's have been successfully studied using monodromy/Galois-theoretic methods. We use these methods to study DP's in general, and analogous situations for inclusions of value sets.

For example, if (f, g) is a SDP, then $f(T) - g(S) \in \mathbb{F}_q[T, S]$ is known to be reducible. This has interesting consequences. We extend this to DP's (that are not pairs of exceptional polynomials) and use reducibility to study the relationship between DP's and SDP's when f is indecomposable. Additionally, we show that if (f, g) is a DP, then $(\deg f, q^t - 1) = (\deg g, q^t - 1)$ for all sufficiently large q^t with $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$. This extends Lenstra's theorem (Carlitz-Wan conjecture) concerning exceptional polynomials.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, and let p denote its characteristic. For any $f \in \mathbb{F}_q[T]$ and finite extension $\mathbb{F}_{q^t}$ of $\mathbb{F}_q$, define the *value set* $\mathcal{V}_f(\mathbb{F}_{q^t})$ to be $\{f(a) \mid a \in \mathbb{F}_{q^t}\}$. Call (f, g) a *Davenport pair* over $\mathbb{F}_q$ if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ for *infinitely* many values of t. For brevity, we use the acronym DP. We will see that (f, g) is automatically a Davenport pair (DP) if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ for one sufficiently large value of t. Call (f, g) a *strong* Davenport pair (SDP) over $\mathbb{F}_q$ if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ for all $t \geq 1$.

Davenport pairs are named in honor of H. Davenport, who, in the 1960's, was interested in a characteristic zero analogue of what we call SDP's. He asked which pairs $(f, g) \in \mathbb{Q}[T]$ have value sets mod l which are equal for almost all primes l. (See Section 3.2 below for more details.)

Call $f \in \mathbb{F}_q[T]$ an *exceptional polynomial* if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t}$ for infinitely many values of t. So f is exceptional if and only if (f, T) is a DP. Thus both SDP's and exceptional polynomials are special types of Davenport pairs. We are interested in the relationship between Davenport pairs on the one hand, and strong Davenport pairs and exceptional polynomials on the other.

For example, if (f, g) is a SDP and (h_1, h_2) is a pair of exceptional polynomials, then $(f \circ h_1, g \circ h_2)$ is a DP. Such DP's have equal value sets over the base field $\mathbb{F}_q$, a property not possessed by all DP's. However, consider the following.

Question 1.1. Suppose (f, g) is a DP over $\mathbb{F}_q$ where

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(1.1) q is sufficiently large and $\mathcal{V}_f(\mathbb{F}_q) = \mathcal{V}_g(\mathbb{F}_q)$.

When is there a pair of exceptional polynomials (h_1, h_2) and a SDP (f', g') such that $f = f' \circ h_1$ and $g = g' \circ h_2$?

Remark 1.2. In the above question, *sufficiently large* means that q is larger than a bound depending on the degrees of f and g. Later we introduce the set $\mathcal{D}_{f,g}$ which allows us to replace (1.1) above with the more natural hypothesis $1 \in \mathcal{D}_{f,g}$ (making the question meaningful even for small q).

We now describe the main results of this paper and their relationships to the *decomposition hypothesis*, the heuristic principle that all DP's satisfying (1.1) have the decomposition described in the above question, or at least behave as if they did.

If (f, g) is a SDP with $\deg f > 1$, then $f(T) - g(S) \in \mathbb{F}_q[S, T]$ is reducible. It follows that $f \circ h_1(T) - g \circ h_2(S) \in \mathbb{F}_q[S, T]$ is also reducible for any pair (h_1, h_2). So the decomposition hypothesis suggests the following result (Theorem 4.8 and Corollary 4.12). *If (f, g) is a DP satisfing (1.1), and f is not an exceptional polynomial, then $f(T) - g(S) \in \mathbb{F}_q[S, T]$ reducible over $\mathbb{F}_q$.*

As another example, consider the following theorem of Lenstra [CF95], conjectured by Carlitz and Wan. *If $f \in \mathbb{F}_q[T]$ is exceptional, then* $\gcd(\deg f, q - 1) = 1$. It is also known that if (f, g) is a SDP, and if the degrees of f and g are prime to the characteristic p, then $\deg f = \deg g$. Thus if $f = f' \circ h_1$ and $g = g' \circ h_2$ where (f', g') is a SDP, (h_1, h_2) is a pair of exceptional polynomials, and $\deg f$ and $\deg g$ are prime to p, then $\gcd(\deg f, q - 1) = \gcd(\deg g, q - 1)$. The decomposition hypothesis suggests that this holds for all DP's satisfying (1.1). This turns out to be true; it is a consequence of our Theorem 5.4 (which is stronger since it makes no assumption on the degrees of f and g).

Finally, consider our Theorem 8.1, a result consistent with the decomposition hypothesis. *Suppose that (f, g) is a DP, and that f is indecomposable. Suppose also that f has degree prime to the characteristic p, and is neither an exceptional polynomial nor linearly related to a cyclic polynomial. Then $g = g' \circ h$ for some SDP (f, g').*

We end this introduction with more questions related to DP's.

Question 1.3. If (h_1, h_2) is a pair of polynomials such that $(f \circ h_1, g \circ h_2)$ is a DP for all SDP's (f, g), must h_1 and h_2 be exceptional polynomials?

Other questions involve *multiplicities* of values. Call (f, g) a *DP with multiplicity* if there are an infinite number of values of t such that f and g not only have the same value sets over $\mathbb{F}_{q^t}$, but the values occur with the same multiplicities. In other words, $f(T) - b$ and $g(T) - b$ have the same number of roots in $\mathbb{F}_{q^t}$ for each $b \in \mathbb{F}_{q^t}$. Similarly, call (f, g) a *SDP with multiplicity* if the multiplicity condition occurs for all values of t.

Question 1.4. Are there SDP's which are not SDP's with multiplicity? Are there DP's which are not DP's with multiplicity?

The characteristic zero analogue of the first part of this question has been considered. In [Mül98, Conjecture 5.2], Muller conjectures a negative answer (pairs of *Kronecker conjugate* polynomials are conjectured to be *arithmetically equivalent*).

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1.1. **The Broader Context.** A polynomial $f \in \mathbb{F}_q[T]$ gives an algebraic map $f : \mathbb{A}^1 \to \mathbb{A}^1$, or, by adding points at infinity, an algebraic map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Our approach, via the arithmetic and geometric monodromy groups associated with the map f, or pairs of maps (f, g), extends considerably to include maps between algebraic curves defined over $\mathbb{F}_q$, and even to finite maps between higher-dimensional varieties. However, we concentrate on polynomial maps. More can be proven for such maps since they have a totally ramified point, infinity, and the maps are between curves of genus zero. Also, limiting the exposition to polynomials contributes to its clarity. However, the reader should be aware of the possibilities, and the challenges of extending beyond polynomial maps.

Some readers may also be interested in the link between Davenport pairs and such topics as Weil vectors, Galois stratification, and Chow motives. Here, a *Weil vector* is the sequence of coefficients of a Poincaré series associated to a number-theoretic counting problem. For example, if V is a projective variety over $\mathbb{F}_q$, we get the familiar Weil vector $\mathcal{N} = (N_1, N_2, \dots)$ where N_t is the number of $\mathbb{F}_{q^t}$-rational points of V. The associated Poincaré series is $P_V(X) = \sum_{t=1}^{\infty} N_t X^t$, and the associated zeta function is $Z_V(X) = \exp\left(\sum_{t=1}^{\infty} N_t X^t/t\right)$.

Weil vectors also arise from other counting problems. For example, let V be a scheme (reduced, separated) of finite type over $\mathbb{Z}$. Consider the Weil vector $\mathcal{N} = (N_1, N_2, \dots)$ where N_t is the number of $\mathbb{Z}/p^t$-rational points which lift to $\mathbb{Z}_p$-rational points. The rationality of the associated Poincaré series was established by Denef [Den84].

Galois stratification is a tool for producing Weil vectors in a wide variety of counting problems (see [FS76] and [FJ86]). Denef and Loeser ([DL]) have recently shown that there is a vital link between Galois stratification and *Chow motives.* Given two Weil vectors $\mathcal{N} = (N_1, N_2, \dots)$ and $\mathcal{N}' = (N_1', N_2', \dots)$, the *characteristic set* $\chi(\mathcal{N}, \mathcal{N}')$ is $\{t \in \mathbb{N}^+ | N_t = N_t'\}$. Such characteristic sets, when the Weil vectors arise from Galois stratification, form Frobenius progressions (Definition 4.5).

To consider the link between DP's and these topics, consider your favorite equation $\Phi(T, \mathbf{U}) = 0$ where $\Phi \in \mathbb{F}_q[T, \mathbf{U}]$ and $\mathbf{U} = (U_1, \dots, U_s)$. Consider also the Weil vector $\mathcal{N}(\Phi) = (N_1(\Phi), N_2(\Phi), \dots)$ where $N_t(\Phi)$ is the number of solutions over $\mathbb{F}_{q^t}$. You often substitute a polynomial or rational function $f(T)$ for T to get the related equation $\Phi(f(T), \mathbf{U}) = 0$. Write Φ_f for $\Phi(f(T), \mathbf{U})$. Let (f, g) be a pair of polynomials, and let $\chi(f, g)$ be the set of t with the property that $\mathcal{V}_f(\mathbb{F}_{q^t})$ and $\mathcal{V}_g(\mathbb{F}_{q^t})$ are equal, and every value occurs with the same multiplicity. We assume $\chi(f, g)$ is infinite. In other words, (f, g) is a DP with multiplicity. Observe that $\chi(f, g) \subseteq \chi(\mathcal{N}(\Phi_f), \mathcal{N}(\Phi_g))$.

This gives us a procedure for generating non-trivial (i.e., non-finite) characteristic sets relating many different pairs of Weil vectors. The resulting characteristic sets must contain a common Frobenius progression $\chi(f, g)$ regardless of your choice of *favorite equation.* This suggests the importance of the the study of Frobenius progressions of the form $\chi(f, g)$ from the more general Weil vector viewpoint.

To study the collection of such sets $\chi(f, g)$, one might wish to generate infinite subsets of the same form using, say, exceptional polynomials. If $h \in \mathbb{F}_q[T]$ is exceptional, let E_h be the set of t where $h : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is bijective ($E_h = \chi(h, T)$ where T is the identity polynomial). Suppose (f, g) is a DP and h_1 and h_2 are exceptional. Then, $\chi(f \circ h_1, g \circ h_2)$ contains $E_{h_1} \cap E_{h_2} \cap \chi(f, g)$. You must know something about $E_{h_1}, E_{h_2}, \chi(f, g)$ to say their intersection is infinite. We see later that $E_{h_1} \cap E_{h_2} \cap \chi(f,g)$ is automatically infinite if f and g are indecomposable with degree prime to p and not linearly related to cyclic polynomials (Theorem 8.1).

2. Notations and conventions

We note that $(f(T), f(T^p))$ is a SDP (as above, $f \in \mathbb{F}_q[T]$, and p is the characteristic of $\mathbb{F}_q$). So, for value set problems, it is harmless to replace any polynomial of the form $f(T^p)$ by $f(T)$. By repeating this process starting with a given polynomial, we obtain a polynomial whose derivative is not the zero polynomial, and whose value set, in all finite extensions, is the same as the original polynomial. This justifies the following convention: *all polynomials appearing in this paper will be assumed to have non-zero derivatives.*

Let F be a field. We are most interested in $F = \mathbb{F}_q$ especially when we are considering value sets, but many of our results hold for more general F. Fix an algebraic closure $\overline{F(z)}$ of $F(z)$ where z is a fixed transcendental element over F, and regard $\overline{F}$ as a subfield of $\overline{F(z)}$. We use the letter T (as above) for a general transcendental element not in $\overline{F(z)}$. We use S and T when we need two independent transcendental elements (neither in $\overline{F(z)}$).

For any $f \in F[T]$, let $\Omega_f \subseteq \overline{F(z)}$ be the splitting field of $f(T)-z$. Since $f(T)-z$ has z-degree 1, it is irreducible in $F(z)[T]$. It is also separable (the derivative f' is assumed not to be the zero polynomial). Call $\widehat{G}_f = \mathrm{Gal}(\Omega_f/F(z))$ the *arithmetic monodromy group* of f. Let $\widehat{F}_f = \Omega_f \cap \overline{F}$. Call $G_f = \mathrm{Gal}(\Omega_f/\widehat{F}_f(z)) \subseteq \widehat{G}_f$ the *geometric monodromy group.* Let $n = \deg(f)$, and let $\{x_1, x_2, \dots, x_n\}$ be the roots of $f(T)-z$ in Ω_f. If H is $\widehat{G}_f$ or a subgroup, denote the elements of H which fix x_i by $H(x_i)$. For example, $\widehat{G}_f(x_i) = \mathrm{Gal}(\Omega_f/F(x_i))$.

Think of $f \in F[T]$ as an algebraic map $f : \mathbb{A}^1 \to \mathbb{A}^1$. By adding a point at infinity, also regard a polynomial (or rational function) as an algebraic covering map $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Now consider the case $F = \mathbb{F}_q$. Here we abuse notation and write $\widehat{\mathbb{F}}_f$ for $\widehat{F}_f$. The quotient $\widehat{G}_f/G_f$ is isomorphic to the cyclic group $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ where $d = [\widehat{\mathbb{F}}_f : \mathbb{F}_q]$. Not only is $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ cyclic, but it is *canonically* isomorphic to $\mathbb{Z}/d$ by the map sending the Frobenius automorphism $a \mapsto a^q$ to 1. Let $\widehat{G}_{f,t}$ be the G_f-coset of elements $\sigma \in \widehat{G}_f$ for which $\sigma|_{\widehat{\mathbb{F}}_f}$ is the map $a \mapsto a^{q^t}$. So $\widehat{G}_{f,t}$ consists of elements of $\widehat{G}_f$ whose image in $\mathbb{Z}/d$ is congruent to t. Thus $\widehat{G}_{f,t}$ depends only on t modulo d.

Now consider analogous definitions for pairs of polynomials (f,g), first for a general field F. Let $\Omega_{f,g} = \Omega_f \cdot \Omega_g \subseteq \overline{F(z)}$ be the splitting field of the product $(f(T)-z)(g(T)-z)$. Let $\widehat{F}_{f,g} = \Omega_{f,g} \cap \overline{F}$. Define the arithmetic monodromy group of the pair as $\widehat{G}_{f,g} = \mathrm{Gal}(\Omega_{f,g}/F(z))$ and the geometric monodromy group as $G_{f,g} = \mathrm{Gal}(\Omega_{f,g}/\widehat{F}_{f,g}(z))$.

Let $\{x_1, x_2, \dots, x_n\}$ be the roots of $f(T)-z$, and $\{y_1, y_2, \dots, y_m\}$ be the roots of $g(T)-z$. Then $\widehat{G}_{f,g}$ acts on $\{x_i\}$, $\{y_j\}$, and the cartesian product $\{x_i\} \times \{y_j\}$. For H equal to $\widehat{G}_{f,g}$ or a subgroup, $H(x_i)$, $H(y_j)$, and $H(x_i, y_j)$ have the usual meanings as stabilizer subgroups.

Note that $\widehat{G}_{f,g}$ is the fiber product of $\widehat{G}_f$ and $\widehat{G}_g$ over the common quotient group $\mathrm{Gal}(\Omega_f \cap \Omega_g/F(z))$.

Now consider the case $F = \mathbb{F}_q$. We abuse notation and write $\widehat{\mathbb{F}}_{f,g}$ for $\widehat{F}_{f,g}$. As before, we have the exact sequence

$$1 \to G_{f,g} \to \widehat{G}_{f,g} \to \mathbb{Z}/d \to 1$$

where $d = [\widehat{\mathbb{F}}_{f,g} : \mathbb{F}_q]$. Define $\widehat{G}_{f,g,t}$ as the elements of $\widehat{G}_{f,g}$ mapping to $t \mod d$. So $\widehat{G}_{f,g,t}$ is the $G_{f,g}$-coset of all σ whose restriction to $\mathbb{F}_{q^d}$ is the automorphism $x \mapsto x^{q^t}$.

We again consider a general field F. Call $f \in F[T]$ *decomposable* over F if $f = f_1 \circ f_2$ with $f_1, f_2 \in F[T]$, $\deg(f_i) > 1$, $i = 1, 2$. Otherwise, f is *indecomposable* over F.

If $f, l_1, l_2 \in F[T]$ are polynomials with $\deg l_1 = \deg l_2 = 1$, then we say f and $l_1 \circ f \circ l_2$ are *linearly related over* F. Linearly related polynomials have isomorphic monodromy groups and equivalent actions of their monodromy groups on their respective root sets.

When comparing value sets, we are interested in a special type of linearly related polynomial pairs. If $f, l \in F[T]$ are polynomials such that $\deg l = 1$, then we say that f and $f \circ l$ are *linearly related on the inside over* F. For example, a pair of polynomials $f, g \in \mathbb{F}_q[T]$ linearly related on the inside over $\mathbb{F}_q$ clearly forms a SDP. We call such SDP's *trivial*. As explained in the next section, there are examples of nontrivial SDP's.

If n is a positive integer then we consider the statement n *is prime to the characteristic of* F to be vacuously true if F has characteristic zero.

3. Review of Earlier Results

We summarize some of what is known concerning value sets, exceptional polynomials, SDP's, and DP's.

3.1. Value Sets from the Monodromy Point of View. Consider a polynomial map as a covering map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Suppose $b \in \mathbb{F}_{q^t} = \mathbb{A}^1(\mathbb{F}_{q^t})$ is not a branch point for this map. Then $b \in \mathcal{V}_f(\mathbb{F}_{q^t})$ if and only if the associated Frobenius element $\mathrm{Frob_t}(b) \in \widehat{G}_f$ fixes at least one root of $f(T) - z$. Further, the number of $a \in \mathbb{F}_{q^t}$ satisfying $f(a) = b$ is equal to the number of fixed points of $\mathrm{Frob_t}(b)$ acting on the roots $\{x_i\}$. We call this fact the *Frobenius Principle*, and note that it follows from an early result of Artin ([Art23], Section 2).

Here $\mathrm{Frob_t}(b)$ is $i\left(\frac{\Omega_f \cdot \mathbb{F}_{q^t}/\mathbb{F}_{q^t}(z)}{P_b}\right)$ where P_b is the place of $\mathbb{F}_{q^t}(z)$ associated to $b \in \mathbb{A}^1(\mathbb{F}_{q^t})$, $\left(\frac{L/K}{P}\right)$ is the Artin symbol, and $i : \mathrm{Gal}(\Omega_f \cdot \mathbb{F}_{q^t}/\mathbb{F}_{q^t}(z)) \to \widehat{G}_f$ is the natural inclusion induced by restriction. The Artin symbol is defined up to conjugacy, so the number of fixed points of $\mathrm{Frob_t}(b)$ is well-defined.

Observe that $\mathrm{Frob_t}(b) \in \widehat{G}_{f,t}$. Conversely, the Chebotarev Density Theorem implies that the proportion of $b \in \mathbb{F}_{q^t}$ with $\mathrm{Frob_t}(b)$ in a given conjugacy class $C \subseteq \widehat{G}_{f,t}$ is approximately proportional to the size of C. More precisely, if $p(C)$ is the proportion of $b \in \mathbb{F}_{q^t}$ with $\mathrm{Frob}_t(b) \in C$, b not a branch point, then

$$\left| p(C) - \frac{|C|}{|\widehat{G}_{f,t}|} \right| < B|C|q^{-t/2}.$$

The best B depends on f, but we can find a B depending only on $n = \deg f$. For example, the bound of Proposition 5.16 of [FJ86], specialized to the current situation, gives $B = 4(g+2)$ where g is the genus of Ω_f. There is a bound in n for this genus g, and hence B. (By Riemann-Hurwitz, bounding g is related to bounding the different divisor for $\Omega_f/\widehat{\mathbb{F}}_f(z)$, which in turn is related to bounding the sizes of higher ramification groups. Use the corollary to Proposition 4, Chapter IV, §1, of [Ser79], together with obvious bounds on different divisor for $\widehat{\mathbb{F}}_f(x_i)/\widehat{\mathbb{F}}_f(z)$, to bound the number of non-trivial higher ramification groups in the geometric monodromy group G_f.)

For $\sigma \in \widehat{G}_{f,t}$, let $N(\sigma)$ be the number elements of $\{x_1, \dots, x_n\}$ fixed by σ. Then

$$\sum_{\sigma \in \widehat{G}_{f,t}} N(\sigma) = \left|\widehat{G}_{f,t}\right|. \tag{3.1}$$

Although this is a corollary of the the Chebotarev Density Theorem (by considering $t' \equiv t \mod d$, where $d = [\widehat{\mathbb{F}}_f : \mathbb{F}_q]$ and t' is large), one can also view it as a consequence of the following group-theoretical lemma (see, e.g., [GW97], Lemma 3.1) taking $H = G_f$, $C = \widehat{G}_{f,t}$, $G \subseteq \widehat{G}_f$ the group generated by G_f and $\widehat{G}_{f,t}$, and $r = 1$.

Lemma 3.1. *Let G be a finite group acting on a finite set S. Let H be a normal subgroup of G such that G/H is cyclic. Finally, let C be a coset whose image generates G/H. Then*

$$\frac{1}{|C|} \sum_{\sigma \in C} N(\sigma) = r$$

where r is the number of H-orbits in S which are also G-orbits, and where $N(\sigma)$ is the number of S-fixed points of $\sigma \in G$.

From (3.1), the following are equivalent:

(3.2) Every element of $\widehat{G}_{f,t}$ fixes at least one element of $\{x_i\}$.
(3.3) Every element of $\widehat{G}_{f,t}$ fixes at most one element of $\{x_i\}$.
(3.4) Every element of $\widehat{G}_{f,t}$ fixes exactly one element of $\{x_i\}$.

Remark 3.2. If any of the above conditions hold, then for all non-branch points $b \in \mathbb{F}_{q^t}$ exactly one root is fixed by $\mathrm{Frob}_t(b)$. So, by the Frobenius Principle, $f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is bijective, at least on the set of points mapping to non-branch points.

When $b \in \mathbb{F}_{q^t}$ is a branch point, one has a *Frobenius coset* instead of a Frobenius element. To determine the number of $a \in \mathbb{F}_{q^t}$ satisfying $f(a) = b$, consider the action of the associated decomposition group D *and* inertia group I on the roots $\{x_i\}$. It is well-known that one counts I-orbits which are also D-orbits ([vdW35]). Using Lemma 3.1 above with $G = D$, $H = I$, and C the Frobenius coset, we see that *the number of $a \in \mathbb{F}_{q^t}$ mapping to b is the average number of $\{x_i\}$ fixed by σ as σ varies over the Frobenius coset.* We call this fact the *Strong Frobenius Principle.* In particular, (3.4) implies bijectivity even when we allow points above branch points. (Note: in the Frobenius Principle or the Strong Frobenius Principle, we can replace $\widehat{G}_f$ with the Galois group of any normal extension of $\mathbb{F}_q(z)$ containing Ω_f).

Definition 3.3. Let $0 \le \epsilon \le 1$. The polynomial map $f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is said to be *ϵ-almost injective* if the proportion of points $b \in \mathbb{F}_{q^t}$ which either have at most one $a \in \mathbb{F}_{q^t}$ satisfying $f(a) = b$ or are branch points is at least $1 - \epsilon$. Similarly, $f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is said to be *ϵ-almost surjective* if the proportion of points $b \in \mathbb{F}_{q^t}$ which are either in the value set $\mathcal{V}_f(\mathbb{F}_{q^t})$ or are branch points is at least $1 - \epsilon$.

The above considerations lead easily to the following theorem.

Theorem 3.4. *Let $0 \le \epsilon < 1/|\widehat{G}_{f,t}|$, and let $\delta = 1/|\widehat{G}_{f,t}| - \epsilon$. If $q^t \ge (B/\delta)^2$ where B is the constant in the Chebotarev Density Theorem, then the following are equivalent.*

(3.5) *$f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is ϵ-almost surjective.*
(3.6) *$f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is ϵ-almost injective.*
(3.7) *$f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is bijective.*
(3.8) *Every element of $\widehat{G}_{f,t}$ fixes exactly one root of $f(T) - z$.*

For general q^t, large or small, (3.8) implies (3.7).

Remark 3.5. See [Fri74, Lemma 2 and Theorem 1] for a generalization to multivariable polynomial maps $\mathbb{A}^n \to \mathbb{A}^n$. This theorem has also been generalized [FGS93, p. 186] to covering maps $X \to Y$ between absolutely irreducible curves over $\mathbb{F}_q$. (The statement in [FGS93] is essentially the case where $\epsilon = 0$, but the methods clearly work for small $\epsilon > 0$.)

The upper bound for ϵ in the implication (3.5) $\Rightarrow$ (3.8) can be replaced by $\frac{1}{\deg f}$. With *a priori* restrictions on the monodromy groups involved, one can often do better (see [GW97]).

Corollary 3.6. *A polynomial $f \in \mathbb{F}_q[T]$ is exceptional if and only if any of the equivalent conditions (3.2) to (3.7) hold for a suitable value of t and ϵ.*

If (3.7) holds for t, then it holds for any divisor of t. This yields the following.

Corollary 3.7. *If $f \in \mathbb{F}_q[T]$ is an exceptional polynomial satisfying any of the equivalent conditions (3.2) to (3.4) for $t = t_0$, then these conditions hold for any t satisfying $\gcd(t, d) | \gcd(t_0, d)$ where $d = [\widehat{\mathbb{F}}_f : \mathbb{F}_q]$.*

A similar analysis gives a monodromy interpretation for $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$.

Theorem 3.8. *Let $f, g \in \mathbb{F}_q[T]$. Suppose that for some t,*

(3.9) *every $\sigma \in \widehat{G}_{f,g,t}$ fixes an element of $\{x_i\}$ if and only if it fixes an element of $\{y_j\}$ (as usual, $\{x_i\}$ are the roots of $f(T) - z$ and $\{y_j\}$ are the roots of $g(T) - z$).*

Then $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$.
Conversely, if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ for t sufficiently large, then (3.9) holds.

Remark 3.9. The converse above, which can be proved with the Chebotarev Density Theorem together with the Frobenius Principle, can be strengthened by replacing the hypothesis $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ with an ϵ-almost equality (analogous to Theorem 3.4).

To prove that (3.9) implies $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ one can use the Strong Frobenius Principle (as in Remark 3.2) to cover both branch points and non-branch points. Alternatively, one can use the following argument, a straightforward adaptation to the current situation of the second part of the proof of Lemma 19.27 of [FJ86]. Let $b \in \mathcal{V}_f(\mathbb{F}_{q^t})$, and let $a \in \mathbb{F}_{q^t}$ be a root of $f(T) - b$. Consider the homomorphism $\mathbb{F}_q[x_1] \to \mathbb{F}_{q^t}$ with $x_1 \mapsto a$ (and so $z \mapsto b$). Extend this to a homomorphism $\varphi : R \to \overline{\mathbb{F}_q}$ where R is the integral closure of $\mathbb{F}_q[z]$ in $\Omega_{f,g}$. Let $D(\varphi) \subseteq \widehat{G}_{f,g}(x_1)$ be the decomposition group associated to φ (i.e., the subgroup fixing $\ker \varphi$). A general fact about decomposition groups is that the homomorphism

$D(\varphi) \to \mathrm{Gal}(\mathbb{F}_{q^s}/\mathbb{F}_q(a))$ associated to the residue maps is surjective where $\mathbb{F}_{q^s}$ is the image of φ. Thus there is an element $\tau \in D(\varphi)$ whose image in $\mathrm{Gal}(\mathbb{F}_{q^s}/\mathbb{F}_q(a))$ is the q^t-power Frobenius map $u \mapsto u^{q^t}$. Note that τ fixes x_1, and that $\tau \in \widehat{G}_{f,g,t}$. It follows from (3.9) that τ fixes some y_j. Let $c = \varphi(y_j)$. The image of τ acting on $\mathbb{F}_{q^s}$ fixes c. Thus, $c \in \mathbb{F}_{q^t}$. Since, $g(c) = b$, conclude $b \in \mathcal{V}_g(\mathbb{F}_{q^t})$.

For inclusions of value sets we have the following:

Theorem 3.10. *Let $f, g, \in \mathbb{F}_q[T]$. Suppose that for some t,*

(3.10) *every $\sigma \in \widehat{G}_{f,g,t}$ which fixes an element of $\{x_i\}$ also fixes an element of $\{y_j\}$.*

Then $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$.

Conversely, if $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for t sufficiently large, then (3.10) holds.

Remark 3.11. We can replace (3.10) with the following.

(3.11) Every $\sigma \in \widehat{G}_{f,g,t}(x_1)$ fixes an element of $\{y_j\}$.

3.2. **Strong Davenport Pairs.**

A good reference for SDP's is [Fri99] which uses the following Galois theoretic characterization (a corollary of Theorem 3.8 above).

Corollary 3.12. *The pair (f, g) of polynomials in $\mathbb{F}_q[T]$ is a SDP if and only if*

(3.12) *for every $\sigma \in \widehat{G}_{f,g}$, σ fixes an element of $\{x_i\}$ if and only if σ fixes an element of $\{y_j\}$.*

An analogous result holds for polynomials over number fields ([FJ86, Lemma 19.27] or [Mül98, Theorem 2.3]). In that case, Condition 3.12 is equivalent to to f and g being *Kronecker conjugate* over a number field K, i.e., their value sets are equal modulo all but a finite number of nonzero prime ideals of K.

The following well-known result ([Fri73, Proposition 3], [FJ86, Lemma 19.31], and [Fri99]) will be generalized to DP's below (Corollary 4.12).

Theorem 3.13. *Let $f, g \in \mathbb{F}_q[T]$. If (f, g) is a SDP where $\deg f > 1$, then $f(T) - g(S) \in \mathbb{F}_q[S, T]$ is reducible.*

This gives several immediate corollaries. For example, if f and g have relatively prime degrees, then (f, g) is not a SDP. As another example, if (f, g) is a SDP with each degree at most 3, then (f, g) is a trivial SDP. To see this, just note that reducibility implies the existence of a linear factor, which implies that f and g are linearly related on the inside.

Deeper corollaries exist. For example, when $f : \mathbb{P}^1 \to \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ are tamely ramified, the results in [Fri73] in characteristic 0 are relevant, implying major restriction on the pair (f, g). We will now review these results.

Let K be a number field, and let $f, g \in K[T]$. If f and g are Kronecker conjugate and $\deg f > 1$ then the analogue of Theorem 3.13 holds: $f(T) - g(S)$ is reducible. This highly restricts f and g, especially in the case where f is indecomposable. The reducibility of $f(T) - g(S)$ together with the indecomposibility of f forces the geometric monodromy group of f to be one of a small list, and it forces $\deg f$ to be one of 7, 11, 13, 15, 21, and 31. The fact that (f, g) are Kronecker conjugate also forces $\deg f = \deg g$. This together with the Grothendieck Lifting Theorem gives the following theorem in positive characteristic.

Theorem 3.14. *Consider a SDP (f, g) over $\mathbb{F}_q$ with the following properties:*

(3.13) *$f : \mathbb{P}^1 \to \mathbb{P}^1$ is tamely ramified.*
(3.14) *f is indecomposable.*

Then $\deg f = \deg g$, *and both* $\deg f$ *and* G_f *are restricted as above.*

The case where $f : \mathbb{P}^1 \to \mathbb{P}^1$ has wild ramification does not result in an upper bound on the degree of f. In fact, we have the following.

Theorem 3.15 ([Fri99], Thm 5.7)**.** *Over any field $\mathbb{F}_q$ there are infinitely many n prime to p for which nontrivial SDP's (f, g) exist with $n = \deg f = \deg g$ and f indecomposable.*

The monodromy groups appearing in these examples are subgroups of the projective linear groups over finite fields of characteristic p.

Note that if (f, g) is a SDP, then $(h \circ f, h \circ g)$ is also a SDP for all $h \in \mathbb{F}_q[T]$. More surprisingly, there are pairs (f, g) which are not SDP's, and $h \in \mathbb{F}_q[T]$ of positive degree such that $(h \circ f, h \circ g)$ is a SDP. Müller [Mül98, §4] gave examples of this over number fields and they apply over suitable $\mathbb{F}_q$. So, there are many non-trivial SDP's. Müller points out (in [Mül98, §5]) that his examples give polynomials with equivalent permutation characters, so they yield SDP's *with multiplicity*.

Finally we mention what is known concerning Davenport's original question. If $K = \mathbb{Q}$ there are no nontrivial Kronecker conjugate polynomials ([Fri73]) with f indecomposable, or with f and g each compositions of two indecomposable polynomials of degree at least 2 ([Mül98]). However, $f(T) = T^8$ and $g(T) = 16T^8$ are Kronecker conjugate polynomials which are the composition of *three* indecomposable polynomials. See [Mül98] for more information.

3.3. **Work Related to Davenport Pairs.** Earlier work related to DP's (which are not assumed *a priori* to be SDP's or exceptional polynomials) are not concerned with DP's *per se*, but study pairs with equal value sets over the ground field $\mathbb{F}_q$. However, when q is large, such pairs are DP's (Corollary 4.2). For example, [Coh81] studies pairs of rational functions $f, g \in \mathbb{F}_q(T)$ satisfying $\mathcal{V}_f(\mathbb{F}_q) \subseteq \mathcal{V}_g(\mathbb{F}_q)$. The main result is a classification of such f and g with $\deg g \le 4$, where the characteristic is greater than 3 and q assumed large (with bounds depending on $\deg f$). There has also been other, much earlier work, by McCann and Williams (value set equalities for polynomials of degree 3), Mordell (also for degree 3), and Carlitz (value set inclusions with $g(T) = T^m$).

Finally, [Ait98] studies the overlap between $\mathcal{V}_f(\mathbb{F}_q)$ and $\mathcal{V}_g(\mathbb{F}_q)$ when the two sets are not equal, which, for large q, yields a criterion for whether or not two polynomials form a DP.

4. Basic Results Concerning Davenport Pairs

Let $f, g \in \mathbb{F}_q[T]$, and let $d = [\widehat{\mathbb{F}}_{f,g} : \mathbb{F}_q]$. Below are corollaries of Theorem 3.8.

Corollary 4.1. *(f, g) is a DP if and only if, for some t, (3.9) holds.*

Corollary 4.2. *(f, g) is a DP if and only if for a sufficiently large t, $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$.*

Here, *sufficiently large* means that q^t is greater than a bound which can be chosen to depend only on the maximum of the degrees of f and g.

Condition (3.9) depends only on $t \mod d$. Thus, if (3.9) holds for one t, it holds for infinitely many t; the set of such t forms a union of arithmetic progressions. For any integer t, denote its image in $\mathbb{Z}/d$ by $\bar{t}$.

Definition 4.3. Let $\mathcal{D}_{f,g} = \{\bar{t} \in \mathbb{Z}/d \mid (3.9) \textit{ holds for } \mathrm{t}\}$. So (f, g) is a DP if and only if $\mathcal{D}_{f,g}$ is not empty, and (f, g) is a SDP if and only if $\mathcal{D}_{f,g} = \mathbb{Z}/d$.

Corollary 4.4. *For t sufficiently large, $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ if and only if $\bar{t} \in \mathcal{D}_{f,g}$. For all t, large or small, $\bar{t} \in \mathcal{D}_{f,g}$ implies $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$.*

The set $\mathcal{D}_{f,g}$ is an example of a *Frobenius set*:

Definition 4.5. A *Frobenius set (mod d)* is a subset S of $\mathbb{Z}/d$ with the following property. If $a \in S$, then so is ua where u is a unit in $\mathbb{Z}/d$. Equivalently, if a, b have the same order in $\mathbb{Z}/d$, then $a \in S$ if and only if $b \in S$. So S is completely determined by the data (d, D), where D is the set of divisors of d representing the orders in $\mathbb{Z}/d$ of the elements in S.

Call a subset A of $\mathbb{N}^+$ (or $\mathbb{N}$ or $\mathbb{Z}$) a *pure Frobenius progression* if there exists a Frobenius set $S \subseteq \mathbb{Z}/d$ so that $a \in A$ if and only if $\bar{a} \in S$. Finally, call a subset A of $\mathbb{N}^+$ a *Frobenius progression* if it differs from a pure Frobenius progression by only a finite number of elements.

Remark 4.6. If (f, g) is a pair of polynomials, then $\mathcal{D}_{f,g}$ is a Frobenius set. The set of t satisfying (3.9) forms a pure Frobenius progression. Finally, the set of t where $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ is a Frobenius progression (containing the associated pure Frobenius progression).

For exceptional polynomials, the associated Frobenius set has additional structure: if $d_1 \in D$ where D is the set of divisors characterizing the Frobenius set, and k is a positive integer such that $kd_1|d$, then $kd_1 \in D$. This follows from Corollary 3.7. One consequence is that $\mathcal{D}_{f,T}$ contains $(\mathbb{Z}/d)^*$. In particular, $1 \in \mathcal{D}_{f,T}$.

When we require $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ with multiplicity, we also get Frobenius progressions. Later we discuss Frobenius progressions in the context of the reducibility of $f(T) - g(S)$.

To prove reducibility, we use the following lemma, a basic application of the Riemann Hypothesis.

Lemma 4.7. *Suppose $\Phi(S, T) \in \mathbb{F}_q[S, T]$ has A_t irreducible factors over $\mathbb{F}_{q^t}[S, T]$, of which N_t are absolutely irreducible. Then M_t, the number of $\mathbb{F}_{q^t}$-points in the algebraic set $\Phi(S, T) = 0$, is approximately $N_t \cdot q^t$. More precisely, $\left|\frac{M_t}{q^t} - N_t\right| < cq^{-t/2}$ for some constant c which depends only on the total degree of Φ.*

Proof. Factor Φ over $\mathbb{F}_{q^t}[S, T]$ as $\Phi_1 \cdots \Phi_{A_t}$. Rearrange the factors so $\Phi_1, \ldots, \Phi_{N_t}$ are absolutely irreducible. Let M_i be the number of $\mathbb{F}_{q^t}$-points of the variety $\Phi_i = 0$. Bezout's Theorem bounds $|M_t - \Sigma M_i|$. For $i > N_t$, $|M_i|$ is bounded (one can use Bezout's Theorem here as well). For $i \le N_t$ let $\widetilde{X}_i$ be the non-singular projective curve corresponding to the affine curve $\Phi_i = 0$. Let $\widetilde{M}_i$ be the number of $\mathbb{F}_{q^t}$-points on $\widetilde{X}_i$. Then $|M_i - \widetilde{M}_i|$ is bounded. All of these bounds depend on the total degree of Φ, and not q^t. Finally, the Riemann Hypothesis bounds $|\widetilde{M}_i - q^t|$, giving the desired bound for $|M_t - N_t \cdot q^t|$. □

Theorem 4.8. *Suppose $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ for sufficiently large t. Let N_t be the number of absolutely irreducible factors of $f(T) - g(S) \in \mathbb{F}_q[S, T]$ defined over $\mathbb{F}_{q^t}$. Then $N_t \ge 1$. Furthermore, $N_t = 1$ if and only if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t} = \mathcal{V}_g(\mathbb{F}_{q^t})$. (So $N_t = 1$ implies f and g are both exceptional polynomials.)*

Here, t sufficiently large *means that q^t is larger than an effectively computable bound which depends only on* $\deg f$ *and* $\deg g$.

Proof. Let M_t be the number of $\mathbb{F}_{q^t}$-points in the algebraic set $f(T) - g(S) = 0$. $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ implies that $M_t \geq q^t$. Thus $N_t \geq 1$ by Lemma 4.7. If $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t}$, then $M_t = q^t$, so $N_t = 1$.

Now suppose $\mathcal{V}_f(\mathbb{F}_{q^t}) \neq \mathbb{F}_{q^t}$. By Theorem 3.4 there is an $A > 0$ (independent of t) with at least $A \cdot q^t$ elements of $\mathcal{V}_f(\mathbb{F}_{q^t})$ having at least two elements of $\mathbb{F}_{q^t}$ mapping to it under f. This implies $M_t \geq q^t \cdot (A+1)$. Thus $N_t > 1$. □

Remark 4.9. Let $f, g \in F[T]$. Gauss' Lemma implies that the factorization of $f(T) - g(S)$ into irreducibles in $F[S,T]$ gives a factorization of $f(T) - g(y_j)$ into irreducibles in $F(y_j)[T]$ (with all factors having positive T-degree). By basic Galois theory, these irreducible factors of $f(T)-g(y_j)$ over $\mathbb{F}_q(y_j)$ correspond to the orbits of $\{x_i\}$ under the action of $\widehat{G}_{f,g}(y_j)$.

Conclude that the F-irreducible factors of $f(T)-g(S)$ correspond to the orbits of $\{x_i\}$ under the action of $\widehat{G}_{f,g}(y_j)$. Further, if Φ is a factor associated with an orbit O then $|O| = \deg_T \Phi$. Similar statements apply for the $\widehat{G}_{f,g}(x_i)$-action on $\{y_j\}$.

Remark 4.10. In the case $f, g \in \mathbb{F}_q[T]$, let G_t be the subgroup of $\widehat{G}_{f,g}$ generated by elements of $\widehat{G}_{f,g,t} \cup G_{f,g}$. In other words, it is the subgroup generated by $G_{f,g}$ and an element lifting the q^t-power Frobenius automorphism. Since G_t is canonically isomorphic to the Galois group of $\Omega_{f,g}\mathbb{F}_{q^t}$ over $\mathbb{F}_{q^t}(z)$, Remark 4.9 gives a natural correspondence between divisors $\Phi \in \mathbb{F}_{q^t}[S,T]$ of $f(T) - g(S)$ (up to multiplication by constants in $\mathbb{F}_{q^t}^\times$) and subsets $B \subseteq \{y_j\}$ on which $G_t(x_i)$ acts. Also, the divisor Φ is absolutely irreducible if and only if the corresponding subset B is an orbit under the action of $G_{f,g}(x_i)$. A similar statement applies when reversing the roles of $\{y_j\}$ vs. $\{x_j\}$ and S vs. T.

Remark 4.11. Suppose $\Phi \in \mathbb{F}_{q^t}[S,T]$ is a divisor of $f(T) - g(S)$. Since $G_t = G_{d'}$ with $d' = \gcd(d,t)$, the above shows that, up to multiplication by a non-zero constant, $\Phi \in \mathbb{F}_{q^{d'}}[S,T]$.

The above theorem and the above remarks give the following:

Corollary 4.12. *If (f,g) is a DP, and f is not exceptional, then $f(T) - g(S)$ is reducible over $\overline{\mathbb{F}}_q$. In fact, if $\bar{t} \in \mathcal{D}_{f,g}$, then $f(T) - g(S)$ is reducible over $\mathbb{F}_{q^t}$.*

Proof. The first statement is clear. The second statement is clear for t sufficiently large, but the above remarks show that reducibility is not actually a property of t, large or small, but a property of t mod d. □

Remark 4.13. The t such that $f(T) - g(S)$ is reducible over $\mathbb{F}_{q^t}$ form a pure Frobenius progression, with associated Frobenius set a *subgroup* of $\mathbb{Z}/d$. Let (d, D) be the data defining this Frobenius set, where D is a set of divisors of d. Then, in contrast with the Frobenius set of an exceptional polynomial, if $d_1|d_2$ are divisors of d, and $d_2 \in D$, then $d_1 \in D$.

Now we consider the analogous situation for inclusions $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$.

Proposition 4.14. *Let $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for t sufficiently large, and let N_t be the number of absolutely irreducible factors of $f(T) - g(S)$ defined over $\mathbb{F}_{q^t}$. Then $N_t \geq 1$. Furthermore, $N_t = 1$ if and only if g is bijective over $\mathcal{V}_f(\mathbb{F}_{q^t})$ in the sense that every non-branch point $b \in \mathcal{V}_f(q^t)$ has exactly one $a \in \mathbb{F}_{q^t}$ mapping to it under g.*

Proof. Let G_t be as in Remark 4.10. Also from this remark, the number N_t of absolutely irreducible factors of $f(T)-G(S)$ defined over $\mathbb{F}_{q^t}$ equals the number of $G_t(x_1)$-orbits which are also $G_{f,g}(x_1)$-orbits.

Use Lemma 3.1 to count such orbits. Conclude that $N_t = r$ where r is the average number of elements of $\{y_j\}$ fixed by σ, as σ varies in $\widehat{G}_{f,g,t}(x_1)$. By Theorem 3.10, $r \geq 1$, and $r = 1$ if and only if every $\sigma \in \widehat{G}_{f,g,t}$ fixing x_1 fixes exactly one element of $\{y_j\}$. So, by the Frobenius Principle and the Chebotarev Density Theorem, $r = 1$ is equivalent to every non-branch point $b \in \mathcal{V}_f(\mathbb{F}_{q^t})$ being the image of exactly one $a \in \mathbb{F}_{q^t}$ under the map induced by g. □

Remark 4.15. The above generalizes Theorem 4.8 since, if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$, bijectivity of f over $\mathcal{V}_f(\mathbb{F}_{q^t})$ is equivalent to $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t}$ (use Theorem 3.4). In fact, we may view the above proof as an alternate proof of Theorem 4.8.

We end with a generalization of Theorem 3.13.

Proposition 4.16. *If $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for all t, and $\deg g > 1$, then $f(T)-g(S)$ is reducible over $\mathbb{F}_q$.*

Proof. By Remark 4.9, the number of factors of $f(T) - g(S)$ is the number of $\widehat{G}_{f,g}(x_1)$-orbits of $\{y_j\}$. By (3.10), each element of $\widehat{G}_{f,g}(x_1)$ fixes at least one element of $\{y_j\}$. If $\{y_j\}$ has only one $\widehat{G}_{f,g}(x_1)$-orbit, then $\widehat{G}_{f,g}(x_1, y_j)$, as y_j varies, are conjugate subgroups of $\widehat{G}_{f,g}(x_1)$. The conjugates, however, of a proper subgroup of a finite group cannot cover the group. □

Remark 4.17. The topic of reducibility will be continued in Section 7.

5. Behavior at Infinity

Many results above generalize to non-polynomial maps. However, polynomial maps have special properties that yield stronger results. For example, polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ are totally ramified above at least one place of $\mathbb{P}^1$, namely the place at infinity. In this section we study the consequences of total ramification.

We begin with a lemma concerning the special case of tame ramification. It is similar to results in the literature (for example [Mül98, Section 2.2]). However, due to the absence of a convenient reference in the needed generality and for the convenience of the reader, we give a proof.

Lemma 5.1. *Let K be a field with discrete valuation v and associated residue field k. Let L be a degree n separable extension of K with valuation w extending v to L. Let M be the normal closure of L over K, and let ω be a valuation of M extending w with residue field $k(\omega)$. Let $G = \mathrm{Gal}(M/K)$ and $H = \mathrm{Gal}(M/L)$. Suppose (L, w) is tamely and totally ramified over (K, v). Then*

- (5.1) *(M, ω) is tamely ramified over (K, v) and unramified over (L, w).*
- (5.2) *The inertia group $I_\omega \subseteq G$ is cyclic and acts transitively and effectively on G/H, and any generator of I_ω corresponds to an n-cycle in $\mathrm{Perm}(G/H)$.*
- (5.3) *$k(\omega) = k(\zeta_n)$ where ζ_n is a primitive nth root of unity.*
- (5.4) *The decomposition group $G_\omega \subseteq G$ is isomorphic to the semidirect product $\mu_n \rtimes_\varphi \widetilde{G}$ where $\mu_n \subseteq k(\omega)^\times$ is the group of nth roots of unity, $\widetilde{G}$ is $\mathrm{Gal}(k(\omega)/k)$, and $\varphi : \tilde{G} \to \mathrm{Aut}(\mu_n)$ is the natural Galois action on the nth roots of unity.*

(5.5) *The isomorphism $G_\omega \to \mu_n \rtimes_\varphi \widetilde{G}$ can be chosen so that the inertia group I_ω corresponds to μ_n, and $H \cap G_\omega$ corresponds to $\widetilde{G}$.*

Proof. Let $\pi = \pi_L$ be a uniformizer for (L, w), and let $h_1 \in K[T]$ be its minimal, monic polynomial. This polynomial is Eisenstein of degree n (its non-leading coefficients have positive valuation, and its constant term is a uniformizer for (K, v)).

Since (L, w) is totally ramified over (K, v), I_ω acts transitively on G/H. The action is effective (G acts effectively on G/H). The property of tame ramification behaves well under composita, and (L, w) is tamely ramified over (K, v). Thus (M, ω) is also tamely ramified over (K, v). So I_ω is cyclic. It acts transitively and effectively on G/H, so its generator acts as an n-cycle. In particular $|I_\omega| = |G/H| = n$, forcing (M, ω) to be unramified over (L, w).

Let K_v, L_w, and M_ω be the completions associated with (K, v), (L, w), and (M, ω). So G_ω is canonically isomorphic to $\mathrm{Gal}(M_\omega/K_v)$. Since h_1 remains irreducible over K_v, $L_w = K_v(\pi)$ and M_ω is the splitting field of h_1 over K_v. Let $h_2(T) = T^n - \pi_K$ where $\pi_K = -h_1(0)$. Note that π_K is a uniformizer for K_v. Let M' be the splitting field of h_2 over K_v. We show that $M_\omega = M'$.

Let $\beta \in M'$ be a root of h_2, and $\zeta \in M'$ a primitive nth root of unity. Note that $h_3(T) \overset{\mathrm{def}}{=} h_1(\beta T)/\pi_K$ is a monic polynomial in M' with coefficients of non-negative valuation. By Hensel's lemma, all the roots $r_1, \dots, r_n$ of h_3 are in M'. Thus $\{r_i\beta\}$, the roots of h_1, are in M'. Conclude $M_\omega \subseteq M'$.

The roots of h_1 correspond to the roots of h_2 as follows. If α is a root of h_1, then expand α in M' in terms of the uniformizer β as $\alpha = \zeta^i\beta + \textit{higher order terms}$. The correspondence sends α to $\zeta^i\beta$. This correspondence is compatible with the $\mathrm{Gal}(M'/K_v)$ action. Conclude that $M_\omega = M'$.

Clearly $I_\omega = \mathrm{Gal}(M_\omega/K_v(\zeta))$, and so $\mathrm{Gal}(K_v(\zeta)/K_v)$ is canonically isomorphic to $\widetilde{G} = \mathrm{Gal}(k(\omega)/k)$. Conclude that $k(\omega) = k(\zeta_n)$.

Replace β by $\beta\zeta^i$, if necessary, so that π corresponds to β. So $H_\omega = H \cap G_\omega$ is the subgroup of G_ω fixing β, and $L_w = K_v(\beta)$. Clearly $H_\omega \cap I_\omega = 1$, and $|H_\omega| = |\widetilde{G}|$. So, restricting the natural homomorphism $G_\omega \to \widetilde{G}$ gives an isomorphism $H_\omega \to \widetilde{G}$. The inverse isomorphism splits the exact sequence

$$1 \to I_\omega \to G_\omega \to \widetilde{G} \to 1.$$

Thus G_ω is isomorphic to a semi-direct product $I_\omega \rtimes \widetilde{G}$ with an isomorphism which sends H_ω to $\widetilde{G}$.

The rule $\gamma \mapsto \overline{\gamma(\beta)/\beta}$ defines a natural isomorphism $I_\omega \to \mu_n$ ([Frö67, Section 8]), where $a \mapsto \bar{a}$ is the residue map. If $\gamma \mapsto \zeta_n^i$, then clearly $\sigma\gamma\sigma^{-1} \mapsto \tilde{\sigma}(\zeta_n^i)$ where $\tilde{\sigma}$ is the image of σ in $\widetilde{G}$. The result follows. □

Example 5.2. Let $f \in F[T]$ be a polynomial of degree prime to the characteristic of F. Then the hypotheses of Lemma 5.1 are satisfied in the following situation.

$K = F(z)$ with $v = \infty_z$, the place at infinity. (So $k = F$).

$L = F(x_1)$ with $w = \infty_{x_1}$. (Here, x_1 is a fixed root of $f(T) - z$).

$M = \Omega_f$ with ω any place above ∞_{x_1}.

$G = \widehat{G}_f$, and H is the subgroup fixing x_1.

Note that we can identify the roots $\{x_1, \dots, x_n\}$ with G/H where a given root x_j corresponds to the coset of elements sending x_1 to x_j.

Applying the Lemma 5.1 to the above example gives the following.

Corollary 5.3. *Suppose $n = \deg f$ is prime to the characteristic of F. Let $\zeta_n \in F$ be a primitive nth root of unity, and $\mu_n \in F^\times$ the group of all nth roots of unity.*

1. The geometric monodromy group G_f contains an element which acts on $\{x_i\}$ as an n-cycle.

2. The field $\widehat{F}_f$ is a subfield of $F(\zeta_n)$. In particular, if $F = \mathbb{F}_q$ and $q \equiv 1 \mod n$, then $\widehat{\mathbb{F}}_f = \mathbb{F}_q$ and $\widehat{G}_f = G_f$.

3. The arithmetic monodromy group $\widehat{G}_f$ contains a subgroup isomorphic to $\mu_n \rtimes \mathrm{Gal}(F(\mu_n)/F)$, and the geometric monodromy group G_f contains a subgroup isomorphic to $\mu_n \rtimes \mathrm{Gal}(F(\mu_n)/\widehat{F}_f)$.

We now give the main theorem of this section. Here $\mathcal{D}_{f,g} \subseteq \mathbb{Z}/d$ is as in Definition 4.3 and $d = [\widehat{\mathbb{F}}_{f,g} : \mathbb{F}_q]$.

Theorem 5.4. *Let $f, g \in \mathbb{F}_q[T]$ with $n = \deg f$ and $m = \deg g$. If (f, g) is a DP, then $\gcd(n, q^t - 1) = \gcd(m, q^t - 1)$ for all positive t with $\bar{t} \in \mathcal{D}_{f,g}$.*

Proof. Let t be a positive integer with $\bar{t} \in \mathcal{D}_{f,g}$. Write $n = n_0 p^u$ and $m = m_0 p^v$ where n_0 and m_0 are prime to p, the characteristic of $\mathbb{F}_q$. We need to show that $\gcd(n_0, q^t - 1) = \gcd(m_0, q^t - 1)$.

Let ∞_z be the infinite place of $\mathbb{F}_q(z)$, and let K be the completion. Fix a place ω of $\Omega_{f,g}$ above ∞_z. Let $\widehat{G}_\omega \subseteq \widehat{G}_{f,g}$ be the decomposition group associated to ω, and $I \subseteq \widehat{G}_\omega$ the inertia group. Thus $\widehat{G}_\omega$ is canonically isomorphic to $\mathrm{Gal}(\Omega_\omega/K)$ where Ω_ω is the completion of $\Omega_{f,g}$ at ω. Choose an element $\phi_t \in \widehat{G}_\omega$ which induces the automorphism $x \mapsto x^{q^t}$ of the residue field. Since $\mathbb{F}_q(x_1)$ is totally ramified over $\mathbb{F}_q(z)$ at ∞_z, the group I acts transitively on $\{x_i\}$. So, after replacing ϕ_t by $\sigma\phi_t$ for a suitable $\sigma \in I$, we can assume that ϕ_t fixes x_1. Note: $\phi_t \in \widehat{G}_{f,g,t}(x_1)$, so ϕ_t must also fix an element of $\{y_j\}$.

Let $I_1 \subseteq I$ be the first higher ramification group. Thus I_1 is a normal p-Sylow subgroup of I with cyclic quotient. Let $\gamma \in I$ be an element whose image in I/I_1 is a generator.

Let R_x be $\widehat{G}_\omega/I_1\widehat{G}_\omega(x_1)$, and consider the map $\{x_i\} \to R_x$ sending x_i to the coset $\sigma I_1 \widehat{G}_\omega(x_1)$ where $\sigma \in \widehat{G}_\omega$ is chosen so that $\sigma(x_1) = x_i$. The fibers of this map are exactly the I_1 orbits of $\{x_i\}$. Since I_1 is normal in I and I acts transitively on $\{x_i\}$, the I_1 orbits all have the same size, that size is a power of p, and the number of I_1 orbits is prime to p. Since $n = |\{x_i\}|$ is the product of $|R_x|$ and the fiber size, it follows that R_x has n_0 elements, and the fibers have size p^u. Likewise, let R_y be $\widehat{G}_\omega/I_1\widehat{G}_\omega(y_1)$, and consider the corresponding map $\{y_j\} \to R_y$. Conclude that $|R_y| = m_0$ and that the fibers have size p^v.

Let $L_x \subseteq \Omega_\omega$ be the fixed field of $I_1\widehat{G}_\omega(x_1)$, and L_y the fixed field of $I_1\widehat{G}_\omega(y_1)$. Let $M_x \subseteq \Omega_\omega$ be the normal closure of L_x over K, and M_y the normal closure of L_y over K. We can identify R_x with $\mathrm{Gal}(M_x/K)/\mathrm{Gal}(M_x/L_x)$. Conclude that $[L_x : K] = n_0$. Since I acts transitively on R_x, L_x/K is totally and tamely ramified. A similar conclusion holds for L_y/K.

Apply Lemma 5.1 to L_x/K and L_y/K. For example, we can identify R_x with $\mathbb{Z}/n_0$ so that ϕ_t fixes $0 \in \mathbb{Z}/n_0$ and γ acts as the map $c \mapsto c + 1$. Consequently, $\gamma^b\phi_t$ acts on $\mathbb{Z}/n_0$ as the map $c \mapsto q^t c + b$. Identify R_y with $\mathbb{Z}/m_0$ in a similar manner.

Now suppose $a = \gcd(q^t - 1, n_0)$ is not a multiple of $\gcd(q^t - 1, m_0)$. Then $\gamma^a\phi_t$, viewed as $c \mapsto q^t c + a$ modulo n_0, clearly fixes an element of R_x. However, $\gamma^a\phi_t$, viewed as $c \mapsto q^t c + a$ modulo m_0, fixes no element of R_y. Let $\rho \in R_x$ be an element fixed by $\gamma^a \phi_t$, and let x_{i_0} be an element of the fiber of $\{x_i\} \to R_x$. Since fibers of this map are I_1 orbits, there is a $\tau \in I_1$ such that $\tau\gamma^a\phi_t$ fixes x_{i_0}. However, $\tau\gamma^a\phi_t$ and $\gamma^a\phi_t$ act on R_y in the same way. So neither has a fixed point in R_y. Thus, $\tau\gamma^a\phi_t$ fixes no element of $\{y_j\}$, contradicting $\bar{t} \in \mathcal{D}_{f,g}$. Conclude that a is a multiple of $\gcd(q^t - 1, m)$.

Similarly, conclude that $\gcd(q^t - 1, m_0)$ is a multiple of $\gcd(q^t - 1, n_0)$. Therefore, $\gcd(q^t - 1, m_0) = \gcd(q^t - 1, n_0)$. □

Remark 5.5. Although we have adopted the convention that polynomials in this paper have non-zero derivatives, the above theorem (and its corollaries) remain valid for polynomials with zero derivatives.

A corollary is Lenstra's theorem [CF95]:

Corollary 5.6. *Let $f \in \mathbb{F}_q[T]$ with $n = \deg f$. If f is an exceptional polynomial, then $\gcd(n, q - 1) = 1$.*

Proof. Apply the theorem to (f, g) where $g(T) = T$. Take $t = 1$ and recall that $\overline{1} \in \mathcal{D}_{f,g}$ since f is exceptional. □

Corollary 5.7. *Let $f, g \in \mathbb{F}_q[T]$ where $\deg f = n_0 p^u$ and $\deg g = m_0 p^v$ with n_0 and m_0 prime to the characteristic p. If (f, g) is a SDP then $n_0 = m_0$.*

Proof. Let t be the order of q modulo $n_0 m_0$. Thus $n_0 m_0 \mid q^t - 1$. By Theorem 5.4,

$$n_0 = \gcd(q^t - 1, \deg f) = \gcd(q^t - 1, \deg g) = m_0.$$

□

The above Theorem and Corollary easily generalize to the case of value set inclusions:

Proposition 5.8. *Let $f, g \in \mathbb{F}_q[T]$ where $\deg f = n$ and $\deg g = m$. For all t such that (3.10) holds, $\gcd(q^t - 1, m)$ divides $\gcd(q^t - 1, n)$.*

Proposition 5.9. *Let $f, g \in \mathbb{F}_q[T]$ where $\deg f = n_0 p^u$ and $\deg g = m_0 p^v$ with n_0 and m_0 prime to the characteristic p. If $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for all t, then m_0 divides n_0.*

6. Induced Decompositions

In this section we review properties of *induced decompositions*. Most of the results are found in or are extensions of results in [Fri73].

Lemma 6.1. *Let $f, g \in F[T]$ be a pair of polynomials. There is a decomposition $f = f_1 \circ f_2$ with $f_1, f_2 \in F[T]$ having the following properties:*

(6.1) *$F(x_i) \cap \Omega_g = F(f_2(x_i))$ for all x_i in $\{x_1, \dots, x_n\}$.*
(6.2) *$\deg f_2 = 1$ if and only if $\Omega_f \subseteq \Omega_g$.*
(6.3) *For all x_i, $f_2(T) - f_2(x_i)$ is an irreducible polynomial over Ω_g.*

Furthermore, the above properties characterize f_1 and f_2 up to composition with linear polynomials (actually (6.1) suffices). More specifically, if $f = f_1 \circ f_2 = f_1' \circ f_2'$ are two such decompositions, then $f_1' = f_1 \circ l^{-1}$ and $f_2' = l \circ f_2$ where $l \in F[T]$ is a linear polynomial.

Call this decomposition and the analogous decomposition of g the *induced decompositions* associated to the pair (f, g).

Proof. Fix a particular root x_i. By Lüroth's Theorem, $F(x_i)\cap\Omega_g = F(w_i)$ for some $w_i \in F(x_i)$. Adjust the choice of w_i by a suitable linear fractional transformation so that $w_i = f_2(x_i)$ and $z = f_1(w_i)$ for some $f_1, f_2 \in F[T]$. Thus $f = f_1 \circ f_2$. Any other choice w_i' has the form $aw_i + b$ where $a, b \in F$, $a \neq 0$. So f_1 and f_2 are unique up to composition with a linear polynomial. Now let x_j be any element of $\{x_1, \dots, x_n\}$ and let $\sigma \in \widehat{G}_{f,g}$ send x_i to x_j. Then $F(x_j) \cap \Omega_g = F(\sigma(x_i)) \cap \Omega_g = F(f_2(\sigma(x_i))) = F(f_2(x_j))$. So (6.1) holds.

To see (6.2), note that $\deg f_2 = 1$ implies $F(x_i)\cap\Omega_g = F(x_i)$. Thus $F(x_i) \subseteq \Omega_g$, so $\Omega_f \subseteq \Omega_g$. The converse is clear.

To see (6.3), consider $f_2(T) - f_2(x_i)$. By (6.1), this polynomial is defined over Ω_g. We will show it is irreducible by showing that $\mathrm{Gal}(\overline{\Omega}_g/\Omega_g)$ acts transitively on its roots. Any root equals some x_j satisfying $f_2(x_i) = f_2(x_j)$. Let $\sigma \in \mathrm{Gal}\left(\overline{F(z)}/F(z)\right)$ satisfy $\sigma(x_i) = x_j$. Let $\tilde\sigma$ be the restriction of σ to Ω_g. Clearly, $\tilde\sigma(f_2(x_i)) = f_2(x_i)$, so $\tilde\sigma \in \mathrm{Gal}(\Omega_g/\Omega_g \cap F(x_i))$. The restriction map

$$\mathrm{Gal}\left(F(x_i)\cdot\Omega_g/F(x_i)\right) \to \mathrm{Gal}\left(\Omega_g/\Omega_g \cap F(x_i)\right)$$

is an isomorphism. Use this to lift $\tilde\sigma$ to $F(x_i)\cdot\Omega_g$, and then to $\overline{\Omega}_g$ so the lifting τ fixes x_i. Then $\sigma\circ\tau^{-1} \in \mathrm{Gal}(\overline{\Omega}_g/\Omega_g)$ and $\sigma\circ\tau^{-1}(x_i) = \sigma(x_i) = x_j$. □

An important feature of these induced decompositions is that they respect the factorization of $f(T) - g(S)$.

Lemma 6.2. *Suppose $f(T) - g(S)$ is reducible over F, and let $f = f_1 \circ f_2$ be the induced decomposition. Then $f_1(T) - g(S)$ is reducible over F. Moreover, substituting $f_2(T)$ for T into the factorization of $f_1(T) - g(S)$ gives the factorization of $f(T) - g(S)$. In particular,* $\deg f_1 > 1$.

Proof. Fix x_i. As in Remark 4.9, factoring $g(S) - f(T)$ over $F[S,T]$ corresponds to finding the orbits of $\{y_j\}$ under the action of $G_{x_i} \overset{\text{def}}{=} \mathrm{Gal}\left(\overline{F(z)}/F(x_i)\right)$. Similarly, factoring $g(S) - f_1(T)$ over $F[S,T]$ corresponds to finding the orbits of $\{y_j\}$ under the action of $G_{f_2(x_i)} \overset{\text{def}}{=} \mathrm{Gal}\left(\overline{F(z)}/F(f_2(x_i))\right)$.

Decompose $\{y_j\}$ into orbits with both groups. Clearly the $G_{f_2(x_i)}$-orbits contain the G_{x_i}-orbits. We show, in fact, they are equal. Let $\sigma \in G_{f_2(x_i)}$ send y_j to y_k. If σ sends x_i to x_l, then x_i and x_l are both roots of $f_2(T) - f_2(x_i)$. By (6.3), there is a $\tau \in \mathrm{Gal}\left(\overline{F(z)}/\Omega_g\right)$ sending x_i to x_l. Thus $\tau^{-1}\circ\sigma \in G_{x_i}$ sends y_j to y_k.

Let $O \subseteq \{y_j\}$ be such an orbit, $\Phi(S,T)$ the corresponding irreducible factor of $g(S) - f_1(T)$, and $\Phi'(S,T)$ the corresponding irreducible factor of $g(S) - f(T)$. The correspondence of Remark 4.9 yields the equation

$$\prod_{y_j \in O} (S - y_j) = c\,\Phi(S, x_i) = c'\,\Phi'(S, f_2(x_i))$$

for some $c, c' \in F$. Thus $c\,\Phi(S,T) = c'\,\Phi'(S, f_2(T))$. □

Corollary 6.3 ([Fri73], Lemma 7)**.** *Suppose $f(T) - g(S)$ is reducible over F. Then there are decompositions $f = f' \circ f''$ and $g = g' \circ g''$ with $f', f'', g', g'' \in F[T]$ such that (i) $f'(T) - g'(S)$ is reducible, (ii) $\Omega_{f'} = \Omega_{g'}$, and (iii) substituting $f''(T)$ for T and $g''(S)$ for S into the factorization of $f'(T) - g'(S)$ gives the the factorization of $f(T) - g(S)$.*

Furthermore, if either $\deg f'$ *or* $\deg g'$ *is prime to p, then* $\deg f' = \deg g'$.

Proof. To prove this, repeatedly use the previous lemma applied to induced decompositions of f *and* g. (Replace f and g with the outer composites as you go along). Eventually you will obtain f_2 and g_2 of degree 1 which implies that $\Omega_f = \Omega_g$.

Now if $\deg f'$ or $\deg g'$ is prime to p, then the place at infinity is tamely ramified in $\Omega_{f'} = \Omega_{g'}$. Conclude that both $\deg f'$ and $\deg g'$ give the order of the inertia group at infinity, so they are equal. (See Lemma 5.1 above). □

Now we show that induced decompositions behave well in certain types of value set situations. Since we are dealing with value sets, we restrict to $F = \mathbb{F}_q$ for the remainder of this section.

Proposition 6.4. *Suppose* $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ *for all* t. *Let* $f = f_1 \circ f_2$ *be the induced decomposition associated to the pair* (f, g) *and let* $g = g_1 \circ g_2$ *be any decomposition (for example, the induced decomposition). Then* $\mathcal{V}_{f_1}(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_{g_1}(\mathbb{F}_{q^t})$ *for all* t.

Proof. All roots of $f_1(T) - z$ have the form $f_2(x_i)$. By Theorem 3.10, we can show $\mathcal{V}_{f_1}(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for all t by showing that any $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}_q(z)}/\mathbb{F}_q(z))$ which fixes $f_2(x_i)$ must also fix some y_j. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}_q(z)}/\mathbb{F}_q(z))$ fixes $f_2(x_i)$, then $x_l = \sigma(x_i)$ is a root of $f_2(T) - f_2(x_i)$. By (6.3), there is a $\tau \in \mathrm{Gal}\left(\overline{\mathbb{F}_q(z)}/\Omega_g\right)$ sending x_l to x_i. So $\tau \circ \sigma$ fixes x_i, and by hypothesis and Theorem 3.10, it must fix some y_j. Since τ fixes y_j, conclude σ also fixes y_j.

Clearly, $\mathcal{V}_g(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_{g_1}(\mathbb{F}_{q^t})$. □

Corollary 6.5. *Suppose* (f, g) *is a SDP with* $\deg g > 1$*, so (as in Proposition 4.16)* $f(T) - g(S)$ *is reducible. Then the decompositions of Corollary 6.3 can be chosen so that* (f', g') *is a SDP.*

Suppose, instead, $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ *for all* t. *Then the decompositions of Corollary 6.3 can be chosen so that* $\mathcal{V}_{f'}(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_{g'}(\mathbb{F}_{q^t})$ *for all* t.

Proposition 6.6. *Suppose* (f, g) *is a SDP with* $n = \deg f$ *and* $m = \deg g$ *prime to* p. *Then* $\Omega_f = \Omega_g$.

Proof. By Corollary 5.9, $n = m$. Let $f = f_1 \circ f_2$ be the induced decomposition associated to the pair (f, g). By Proposition 6.4, (f_1, g) is also a SDP. By Corollary 5.9 again, $\deg f_1 = m$. Hence, $\deg f_2 = 1$. Thus, by (6.2), $\Omega_f \subseteq \Omega_g$. A similar argument gives the other inclusion. □

Finally, we show that in some circumstances the induced decompositions behave well for DP's.

Proposition 6.7. *Suppose* (f, g) *is a DP with* $\widehat{\mathbb{F}}_{f,g} = \widehat{\mathbb{F}}_g$. *Let* $f = f_1 \circ f_2$ *be the induced decomposition associated to the pair* (f, g). *Then* (f_1, g) *is a DP. Furthermore,* $\mathcal{D}_{f,g} \subseteq \mathcal{D}_{f_1,g}$*, both being subsets of* $\mathbb{Z}/d$ *where* $d = [\widehat{\mathbb{F}}_g : \mathbb{F}_q]$.

Proof. We need to verify (3.9) with (f_1, g) for all t such that $\bar{t} \in \mathcal{D}_{f,g}$ (Definition 4.3). So, let $\sigma \in \widehat{G}_{f_1,g,t}$ where $\bar{t} \in \mathcal{D}_{f,g}$, and let $\tilde{\sigma} \in \widehat{G}_{f,g,t}$ be a element restricting to σ. Note that the roots of $f_1(T) - z$ have the form $f_2(x_i)$.

First, suppose σ fixes y_j. So $\tilde{\sigma}$ fixes y_j, and, by property (3.9), $\tilde{\sigma}$ fixes some x_i. Thus $\tilde{\sigma}$, and hence σ, fix $f_2(x_i)$.

Now suppose σ fixes $f_2(x_i)$. Let $x_l = \tilde{\sigma}(x_i)$ (so x_l is a root of $f_2(T) - f_2(x_i)$). By (6.3) there is a $\tau \in \mathrm{Gal}(\Omega_{f,g}/\Omega_g)$ sending x_i to x_l. Since τ fixes Ω_g, it also fixes $\widehat{\mathbb{F}}_g = \widehat{\mathbb{F}}_{f,g}$. Hence $\tau^{-1} \circ \tilde{\sigma} \in \widehat{G}_{f,g,t}(x_i)$. Since $\bar{t} \in \mathcal{D}_{f,g}$, property (3.9) applies, and $\tau^{-1} \circ \tilde{\sigma}$ must fix some y_j. Since τ fixes $y_j \in \Omega_g$, conclude σ also fixes y_j. □

Remark 6.8. An analogous result holds for inclusions.

7. Reducibility and Representations

This section links the reducibility of $f(T) - g(S)$ to the behaviour of the associated Galois representations. It builds on the characteristic zero results of [Fri73] and the positive characteristic results of [Fri99].

7.1. **Group Theory Lemmas.**

Let G be a finite group acting on a finite set $\mathcal{S} = \{s_i\}$ with N elements. This permutation action of G has an associated linear action of G on a complex vector space $V_{\mathcal{S}}$ as follows. Let $V_{\mathcal{S}}$ be an N-dimensional complex vector space with a chosen basis $(\mathbf{s}_i)$. Have $\sigma \in G$ act on $V_{\mathcal{S}}$ by the unique linear transformation that sends $\mathbf{s}_{i_1}$ to $\mathbf{s}_{i_2}$ if and only if σ (acting on $\mathcal{S}$) sends s_{i_1} to s_{i_2}.

Let $\chi_{\mathcal{S}}$ be the character of the action of G on $V_{\mathcal{S}}$. The following lemma, a special case of Lemma 3.1, is easy and well-known.

Lemma 7.1. *For all $\sigma \in G$, the value of the character $\chi_{\mathcal{S}}(\sigma)$ is the number of elements of $\mathcal{S}$ fixed by σ. Furthermore,*

$$< \chi_{\mathcal{S}}, 1 > = \frac{1}{|G|} \sum_{\sigma \in G} \chi_{\mathcal{S}}(\sigma) \; = \; \textit{number of orbits in } \mathcal{S}.$$

Here we use the standard Hermitian inner product on the vector space $\mathbb{C}^{|G|}$ of functions from G to $\mathbb{C}$:

$$< f_1, f_2 > \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{\sigma \in G} f_1(\sigma) \overline{f_2(\sigma)}.$$

The irreducible characters form an orthogonal basis.

The $\mathbb{C}[G]$-module $V_{\mathcal{S}}$ decomposes as $\mathbf{1}_{\mathcal{S}} \oplus V'_{\mathcal{S}}$ where $\mathbf{1}_{\mathcal{S}}$ is the submodule generated by $\sum_i \mathbf{s}_i$ and where $V'_{\mathcal{S}}$ is the kernel of the augmentation map $\eta : V_{\mathcal{S}} \to \mathbb{C}$, $\sum_i \lambda_i \mathbf{s}_i \mapsto \sum \lambda_i$. Let $\chi'_{\mathcal{S}}$ be the character associated to $V'_{\mathcal{S}}$. In particular, $\chi_{\mathcal{S}} = 1 + \chi'_{\mathcal{S}}$.

Lemma 7.2. *If G acts transitively on $\mathcal{S}$, then $\mathbf{1}_{\mathcal{S}}$ consists of all elements of $V_{\mathcal{S}}$ fixed by G. Furthermore $< 1, \chi'_{\mathcal{S}} >= 0$, and so $< \chi'_{\mathcal{S}}, \chi'_{\mathcal{S}} > = < \chi_{\mathcal{S}}, \chi_{\mathcal{S}} > - 1$.*

Proof. The identity character appears exactly once in the permutation representation of each orbit of G acting on S. So, transitivity means that $\mathbf{1}$ doesn't appear in χ'_S. Apply the inner product of $\mathbf{1} + \chi'_S$ to itself to get the given relation. □

Remark 7.3. In general, $< \chi'_{\mathcal{S}}, \chi'_{\mathcal{S}} > = < \chi_{\mathcal{S}}, \chi_{\mathcal{S}} > - (2r-1)$ where r is the number of orbits in $\mathcal{S}$.

Now let the finite group G act transitively on two finite sets A and B. Consider also the associated G-action on $A \times B$. The following easy lemma is the starting point for our analysis of reducibility.

Lemma 7.4. *The number of orbits of the product $A \times B$ under the action of G is equal to $< \chi_A, \chi_B >$.*

Proof. The character $\chi_{A\times B}$ associated to the action of G on $A \times B$ is just $\chi_A \cdot \chi_B$. By Lemma 7.1, the number of orbits in $A \times B$ is

$$\frac{1}{|G|}\sum_{\sigma\in G}\chi_{A\times B}(\sigma) \;=\; \frac{1}{|G|}\sum_{\sigma\in G}\chi_A(\sigma)\chi_B(\sigma) \;=<\chi_A,\chi_B>.$$

(Note: this proof does not require the transitivity assumption). □

The following well-known characterization of double transitivity is an immediate consequence of the above results.

Corollary 7.5. *Suppose the action of G on $\mathcal{S}$ is transitive where $|\mathcal{S}| \geq 2$. Then the following are equivalent:*

(7.1) *The action of G on $\mathcal{S}$ is doubly transitive.*
(7.2) *There are exactly two orbits in $\mathcal{S}\times\mathcal{S}$ under the action of G.*
(7.3) $<\chi_{\mathcal{S}},\chi_{\mathcal{S}}>=2$.
(7.4) $<\chi'_{\mathcal{S}},\chi'_{\mathcal{S}}>=1$.
(7.5) $V'_{\mathcal{S}}$ *is an irreducible* $\mathbb{C}[G]$*-module.*

Remark 7.6. In this corollary, we can replace they hypothesis that G acts transitively on $\mathcal{S}$ with the alternate hypothesis $|\mathcal{S}| > 2$.

The following is also an easy consequence of Lemma 7.4.

Lemma 7.7. *If G acts doubly transitively on A and $|A| \geq 2$, then the multiplicity of V'_A in the decomposition of V'_B is one less than the number of G-orbits of $A \times B$.*

Corollary 7.8. *Suppose $|A| \geq 2$ and G acts doubly transitively on A. Suppose also that $|A| = |B|$. Then the following are equivalent:*

(7.6) $\chi_A = \chi_B$.
(7.7) *$A \times B$ has more than one orbit.*
(7.8) *$A \times B$ has exactly two orbits.*
(7.9) *V_A and V_B are isomorphic as* $\mathbb{C}[G]$*-modules.*

Remark 7.9. Note that if (7.6) (or its equivalents) hold, then G must act doubly transitively on B as well (by Corollary 7.5).

We refine (7.9) above by explicitly constructing a natural isomorphism from V_A to V_B when (i) G acts doubly transitively on A, (ii) $|A| = |B|$, and (iii) $A \times B$ has more than one orbit. We first define natural maps $V_A \to V_B$ without using these three hypotheses, and then show that when the hypotheses hold, one gets an isomorphism.

First choose a G-invariant subset Γ of $A \times B$, for example a G-orbit. For convenience, label elements: $A = \{a_i\}$, $B = \{b_j\}$. Consider the matrix $E = [\epsilon_{i,j}]$ where $\epsilon_{i,j}$ is 1 if $(a_i, b_j) \in \Gamma$, and 0 otherwise. Consider the linear map $\psi : V_A \to V_B$ defined by the matrix E:

$$\psi(\mathbf{a}_i) \stackrel{\text{def}}{=} \sum_j \epsilon_{i,j}\mathbf{b}_j, \qquad \text{so} \quad \psi\left(\sum_i \lambda_i \mathbf{a}_i\right) = \sum_j\left(\sum_i \lambda_i\epsilon_{i,j}\right)\mathbf{b}_j.$$

Here and below, $(\mathbf{a}_i)$ is the basis of V_A associated to $A = \{a_i\}$, and $(\mathbf{b}_j)$ is the basis of V_B associated to $B = \{b_j\}$.

The following lemma follows directly from the definition (and does not depend on transitivity).

Lemma 7.10. *The map $\psi : V_A \to V_B$ is a $\mathbb{C}[G]$-module morphism.*

Now we investigate some of the consequences of transitivity.

Lemma 7.11. *If Γ is nonempty, then restricting ψ to $\mathbf{1}_A$ gives an isomorphism $\mathbf{1}_A \to \mathbf{1}_B$ of $\mathbb{C}[G]$-modules.*

Proof. Check that ψ sends $\sum_i \mathbf{a}_i$ to $C \sum_j \mathbf{b}_j$ where, for each b_j, $C = C_j$ is the number of $a_i \in A$ with the property that $(a_i, b_j) \in \Gamma$. (C_j is independent of j by transitivity). □

Lemma 7.12. *Restricting ψ to V_A' gives a $\mathbb{C}[G]$-module morphism $\psi' : V_A' \to V_B'$.*

Proof. Check that $\eta_B \circ \psi = D \cdot \eta_A$ where η_A and η_B are the augmentation maps, and, for each a_i, $D = D_i$ is the number of $b_j \in B$ with the property that $(a_i, b_j) \in \Gamma$. (D_i is independent of i by transitivity). □

Lemma 7.13. *If $\Gamma \subseteq A \times B$ is not empty, nor all of $A \times B$, then $\psi' : V_A' \to V_B'$ (defined above) has non-trivial image.*

Proof. Fix a basis vector $\mathbf{a}_i$ of V_A. Since Γ is non-empty, $\epsilon_{i,j_1} = 1$ for some j_1. Since Γ is a proper subset of $A \times B$, $\epsilon_{i,j_2} = 0$ for some j_2. Let $\sigma \in G$ be an element such that $\sigma(b_{j_1}) = b_{j_2}$. Then $\psi'(\sigma(\mathbf{a}_i) - \mathbf{a}_i) \neq 0$. □

Lemma 7.14. *Suppose that G acts doubly transitively on A, and that Γ is not empty nor all of $A \times B$. Then $\psi : V_A \to V_B$ is injective.*

Proof. By Corollary 7.5, V_A' is irreducible, and by the previous lemma, the map $\psi' : V_A' \to V_B'$ is not trivial. Thus ψ' is injective. By Lemma 7.11, the map $\mathbf{1}_A \to \mathbf{1}_B$ induced by ψ is an isomorphism. Thus $\psi : \mathbf{1}_A \oplus V_A' \to \mathbf{1}_B \oplus V_B'$ is injective. □

Proposition 7.15. *Suppose that (i) G acts doubly transitively on A, (ii) $|A| = |B|$, and (iii) Γ is a non-empty proper subset of $A \times B$ invariant under the action of G. Then $\psi : V_A \to V_B$ is an isomorphism.*

Proof. By the previous lemma, ψ is injective. Since V_A and V_B have the same dimension, ψ is an isomorphism. □

Also of interest is the following ([Fri73, Lemma 2]).

Lemma 7.16. *Suppose G acts doubly transitively on A, and $|A| = |B| \geq 2$. Suppose also that, for all $\sigma \in G$, $\chi_A(\sigma) > 0$ if and only if $\chi_B(\sigma) > 0$. Then $\chi_A = \chi_B$.*

Proof. Recall $\chi_A' = \chi_A - 1$ and $\chi_B' = \chi_B - 1$. By hypothesis, for all $\sigma \in G$, $\chi_A'(\sigma) < 0$ if and only if $\chi_B'(\sigma) < 0$. If $\sigma = 1$ then $\chi_A'(\sigma) > 0$ and $\chi_B'(\sigma) > 0$. Thus

$$< \chi_A, \chi_B > = < \chi_A', \chi_B' > + 1 \geq 2.$$

The result follows from Lemma 7.4 and Corollary 7.8. □

Remark 7.17. This shows that if (f, g) is a SDP and if $\widehat{G}_f$ acts doubly transitively on $\{x_i\}$, then (f, g) is actually a SDP with multiplicity. (This can also be seen as a corollary of Proposition 7.26 and Theorem 3.13).

Remark 7.18. In the above lemma, we can replace the hypothesis that *for all $\sigma \in G$, $\chi_A(\sigma) > 0$ if and only if $\chi_B(\sigma) > 0$* with the hypothesis that *for all $\sigma \in G$, $\chi_A(\sigma) \leq 1$ if and only if $\chi_B(\sigma) \leq 1$.*

Finally, we describe one more consequence of double transitivity which will be useful later.

Lemma 7.19. *Let* Γ *be an orbit of* $A \times B$ *where, as above,* G *acts transitively on* A *and* B*. Suppose also that* G *acts doubly transitively on* A *where* $|A| \geq 2$*. Then*

$$|A||B|(|A|-1) \ \mid \ |\Gamma|(|\Gamma|-|B|).$$

Proof. For $b \in B$, let $\Gamma_b \stackrel{\text{def}}{=} \{a|(a,b) \in \Gamma\}$. Note, $k \stackrel{\text{def}}{=} |\Gamma_b|$ is independent of $b \in B$ since G acts transitively on B.

Now consider the set

$$\Gamma' = \{(a, a', b) \mid (a,b), (a',b) \in \Gamma, \text{ and } a \neq a' \}.$$

For distinct elements a, a' of A, let $\Gamma'_{a,a'} \stackrel{\text{def}}{=} \{b|(a,a',b) \in \Gamma'\}$. Note, $l \stackrel{\text{def}}{=} |\Gamma'_{a,a'}|$ is independent of a and a' since G acts doubly transitively on A.

We count the number of element of Γ' in two ways:

$$|\Gamma'| = |A|(|A|-1)l = |B|k(k-1)$$

Now multiply both sides by $|B|$ and use the equation $k|B| = |\Gamma|$. ∎

7.2. **Reducibility.** In this section, unless otherwise stated, F is a general field and $f, g \in F[T]$. Remark 4.9 describes the factorization of $f(T) - g(S)$ in $F[S,T]$ in terms of $\widehat{G}_{f,g}(y_j)$-orbits of $\{x_i\}$. However, there is another natural description of the factorization of $f(T) - g(S)$ in $F[S,T]$ whose validity follows immediately from Remark 4.9:

Proposition 7.20. *Consider the action of* $\widehat{G}_{f,g}$ *on* $\{x_i\} \times \{y_j\}$ *induced by the natural actions of* $\widehat{G}_{f,g}$ *on* $\{x_i\}$ *and* $\{y_j\}$*. The irreducible factors of* $f(T)-g(S)$ *in* $F[S,T]$ *naturally correspond to the orbits of* $\{x_i\}\times\{y_j\}$*. This correspondence sends an irreducible factor* Φ *of* $f(T) - g(S)$ *to the orbit consisting of all pairs* (x_i, y_j) *satisfying* $\Phi(x_i, y_j) = 0$ *in* $\Omega_{f,g}$*.*

Let $O \subseteq \{x_i\} \times \{y_j\}$ *be such an orbit, and let* $\Phi \in F[S,T]$ *be the corresponding factor of* $f(T) - g(S)$*. Then*

$$|O| = \deg f \cdot \deg_S \Phi = \deg g \cdot \deg_T \Phi.$$

Corollary 7.21. *The number of irreducible factors of* $f(T) - g(S)$ *in* $F[S,T]$ *is bounded by* $w = \gcd(\deg f, \deg g)$*. In fact, the* T*-degree of any irreducible factor of* $f(T) - g(S)$ *is a multiple of* $(\deg f)/w$*, and the* S*-degree of any such factor is a multiple of* $(\deg g)/w$*.*

Remark 7.22. The above corollary generalizes the well-known result of Ehrenfeucht that $\gcd(\deg f, \deg g) = 1$ implies $f(T) - g(S)$ is irreducible.

Corollary 7.23. *Let* Φ *be an irreducible divisor of* $f(T)-g(S)$ *in the ring* $F[S,T]$*. If* $\deg f = \deg g$*, then*

$$\deg \Phi = \deg_T \Phi = \deg_S \Phi$$

where the first of these is is the total degree of Φ*.*

Now consider the special case $F = \mathbb{F}_q$. Factoring $f(T)-g(S)$ over $\mathbb{F}_q$ amounts to describing the orbits of $\{x_i\} \times \{y_j\}$ under the action of the arithmetic monodromy group $\widehat{G}_{f,g}$, and factoring $f(T) - g(S)$ over $\overline{\mathbb{F}}_q$ amounts to describing the orbits of $\{x_i\} \times \{y_j\}$ under the action of the geometric mondromy group $G_{f,g}$ (to see the latter, use the canonical isomorphism between $\mathrm{Gal}\big(\Omega_{f,g}\overline{\mathbb{F}}_q/\overline{\mathbb{F}}_q(T)\big)$ and $G_{f,g}$, and then apply Proposition 7.20 with $F = \overline{\mathbb{F}}_q$).

In what follows, let $d = [\widehat{\mathbb{F}}_{f,g} : \mathbb{F}_q]$.

Proposition 7.24. *Let Φ be an irreducible factor of $f(T)-g(S)$ over $\overline{\mathbb{F}}_q[S,T]$, and let (x_{i_0}, y_{j_0}) be in the corresponding orbit under $G_{f,g}$. Then a non-zero constant multiple of Φ is defined over $\mathbb{F}_{q^t}$ if and only if $\bar{t}$ is in the subgroup of $\mathbb{Z}/d$ generated by the image of $\widehat{G}_{f,g}(x_{i_0}, y_{j_0})$ under $\widehat{G}_{f,g} \to \mathbb{Z}/d$.*

Proof. Let G_t consist of the elements in $\widehat{G}_{f,g}$ whose image in $\mathbb{Z}/d$ is in the subgroup generated by $\bar{t}$. Note: G_t is isomorphic to $\mathrm{Gal}\,(\mathbb{F}_{q^t}\Omega_{f,g}/\mathbb{F}_{q^t}(T))$ and the action on $\{x_i\}$ and $\{y_j\}$ are preserved by this isomorphism. Thus, by Proposition 7.20, irreducible factors of $f(T)-g(S)$ in $\mathbb{F}_{q^t}[S,T]$ correspond to G_t-orbits of $\{x_i\}\times\{y_j\}$.

Let $\Phi' \in \mathbb{F}_{q^t}[S,T]$ be the irreducible factor corresponding to the G_t-orbit containing (x_{i_0}, y_{j_0}). The nature of the correspondence in Proposition 7.20 implies that Φ divides Φ' in $\overline{\mathbb{F}}_q[S,T]$. The degrees of Φ and Φ' are determined by the sizes of the associated orbits, so Φ' is a non-zero constant multiple of Φ if and only if these orbits are the same size. This in turn is equivalent to the following:

$$\frac{|G_t|}{|G_t(x_{i_0}, y_{j_0})|} = \frac{|G_{f,g}|}{|G_{f,g}(x_{i_0}, y_{j_0})|}, \quad \text{or equivalently} \quad \frac{d}{a} = \frac{|G_t|}{|G_{f,g}|} = \frac{|G_t(x_{i_0}, y_{j_0})|}{|G_{f,g}(x_{i_0}, y_{j_0})|}$$

where $a = \gcd(d,t)$. The ratio $|G_t(x_{i_0}, y_{j_0})|/|G_{f,g}(x_{i_0}, y_{j_0})|$ determines the image of $G_t(x_{i_0}, y_{j_0})$ in $\mathbb{Z}/d$, and the above equation holds if and only if this image is the subgroup generated by $\bar{t}$. Finally, this occurs if and only if the image of $\widehat{G}_{f,g}(x_{i_0}, y_{j_0})$ in $\mathbb{Z}/d$ contains $\bar{t}$. □

We return to the case of general field F. Let V_f and V_g be the $\mathbb{C}[\widehat{G}_{f,g}]$-modules associated to the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ and $\{y_j\}$ respectively. Let χ_f and χ_g be the associated characters. Lemma 7.4 and Proposition 7.20 give the following.

Proposition 7.25. *The number of irreducible factors of $f(T) - g(S)$ in $F[S,T]$ is equal to $< \chi_f, \chi_g >$.*

Corollary 7.8 and Proposition 7.20 give the following.

Proposition 7.26. *Suppose that the degrees of f and g are equal and greater than one, and that the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ is doubly transitive. Then the following are equivalent.*

(7.10) $\chi_f = \chi_g$.
(7.11) *$f(T) - g(S)$ is reducible in $F[S,T]$.*
(7.12) *$f(T) - g(S)$ factors into exactly two irreducible factors in $F[S,T]$.*
(7.13) *V_f and V_g are isomorphic as $\mathbb{C}[\widehat{G}_{f,g}]$-modules.*

We note that if $F = \mathbb{F}_q$ and $\chi_f = \chi_g$, then Corollary 3.12 (together with the observation in Lemma 7.1) implies that (f,g) is a SDP. Thus we get the following.

Corollary 7.27. *Let $F = \mathbb{F}_q$. Suppose (i) the degrees of f and g are equal, (ii) the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ is doubly transitive, and (iii) $f(T) - g(S)$ is reducible in $\mathbb{F}_q[S,T]$. Then (f,g) is a SDP with multiplicity.*

We can also use Proposition 7.26 to prove the following:

Lemma 7.28. *Suppose $f, g \in F[T]$ are polynomials of degree at least three which are linearly related on the inside over the separable closure F^{sep}. Suppose also that the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ is doubly transitive. Then f and g are linearly related on the inside over F.*

Proof. Let E be a finite Galois extension of F over which f and g are linearly related on the inside. This implies that $f(T) - g(S)$ has a linear factor defined over E. Proposition 7.26 implies $f(T) - g(S)$ has exactly two factors defined over E, one of which is linear and so the other must be of total degree greater than 1. Hence the factors are invariant under the natural $\mathrm{Gal}(E/F)$ action. Since $f(T) - g(S)$ has a linear factor defined over F, the polynomials f and g are linearly related on the inside over F. ☐

Lemma 7.19 gives the following.

Proposition 7.29. *Let Φ be a factor of $f(T) - g(S)$ of total degree $k > 1$ which is irreducible in $F[S, T]$. Suppose that the degrees of f and g are both equal to $n > 1$. Suppose also that the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ is doubly transitive. Then*

$$n - 1 \mid k(k - 1).$$

Proof. Let O be the orbit corresponding to Φ via the correspondence of Proposition 7.20. Note that $\deg_S \Phi = \deg \Phi = k$ by Corollary 7.23. Apply Lemma 7.19 with $A = \{x_i\}$, $B = \{y_j\}$ and $\Gamma = O$. By Proposition 7.20, $|O| = nk$. ☐

Corollary 7.30. *Suppose the degrees of f and g are both equal to $n > 2$, the action of $\widehat{G}_{f,g}$ on $\{x_i\}$ is doubly transitive, and $f(T) - g(S)$ is reducible over F. Then the two irreducible factors of $f(T) - g(S)$ have non-equal degrees.*

Proof. There are exactly two factors by Proposition 7.26. Suppose they both have degree k, i.e., $n = 2k$. Then $n - 1 = 2k - 1$ is prime to k and $k - 1$. Thus $n - 1$ cannot divide $k(k - 1)$, contradicting the previous proposition. ☐

7.3. **Polynomials With Doubly Transitive Monodromy Groups.** Many of the above results (Proposition 7.26 to Corollary 7.30) depend on the double transitivity of monodromy groups. The classification of polynomials with doubly transitive geometric monodromy groups is well-known, at least when the degree is prime to the characteristic. We describe this classification. Throughout this section, let $f \in F[T]$ have degree n at least 2, and let V_f be the associated $\mathbb{C}[G_f]$ module with character χ_f.

Lemma 7.31. *Suppose the arithmetic monodromy group $\widehat{G}_f$ acts doubly transitively on $\{x_i\}$. Then f is indecomposable over F.*

Proof. Suppose that $f = f_1 \circ f_2$ with $f_1, f_2 \in F[T]$ of degrees at least two. Then

$$f(T) - f(S) \;=\; (T - S)\;\Phi_1(f_2(S), f_2(T))\;\Phi_2(S, T)$$

where

$$\Phi_i(S, T) \;\stackrel{\mathrm{def}}{=}\; \frac{f_i(T) - f_i(S)}{T - S}.$$

Thus $f(T) - f(S)$ has at least 3 irreducible factors, contradicting Proposition 7.26. ☐

The following gives a partial converse.

Lemma 7.32. *Suppose $f \in F[T]$ is indecomposable over F with $n = \deg f$ composite and prime to the characteristic of F. Then the arithmetic and geometric mondromy groups of f act doubly transitively on $\{x_i\}$.*

Proof. A theorem of Fried and MacRae implies that, since n is prime to the characteristic of F, f is indecomposable over $\overline{F}$. Thus G_f acts primitively on $\{x_i\}$. By Corollary 5.3, there is an element of G_f which acts as an n-cycle on $\{x_i\}$. Schur proved that a finite group G acting on a set with N elements acts doubly transitively if (i) the action is primitive, (iii) G contains an element acting as an N-cycle, and (iii) N is composite. □

The above lemmas allow us to concentrate on the case n a prime. In the case $n = 2$ the action is trivially doubly transitive, thus we can restrict $\deg f = n$ to odd primes (different from the characteristic of F). Before finishing the classification, we describe important families of polynomials whose geometric monodromy groups *do not* act doubly transitively on $\{x_i\}$.

Consider the cyclic polynomials $f(T) = T^n$. Here G_f is a cyclic group of order n with generator acting on the roots $\{x_i\}$ as an n-cycle. Furthermore, $< \chi_f, \chi_f >= n$ and $f(T) - g(S)$ factors into n linear factors. So when $n > 2$, the action of G_f on $\{x_i\}$ is not doubly transitive.

The other main family of examples are the Chebyshev polynomials:

Definition 7.33. The Chebyshev polynomial τ_n of degree n is defined to be the polynomial in $F[T]$ satisfying

$$\tau_n\left(T + \frac{1}{T}\right) = T^n + \frac{1}{T^n}.$$

The following well-known result is easily verified (the recursion can be used to prove existence).

Lemma 7.34. *For every $n \geq 1$ the nth Chebyshev polynomial τ_n exists and is unique (for any given characteristic). It is monic,*

$$\tau_1(T) = T,$$
$$\tau_2(T) = T^2 - 2, \quad \textit{and}$$
$$\tau_{n+2}(T) = T \cdot \tau_{n+1}(T) - \tau_n(T) \quad \textit{for all } n \geq 1.$$

Remark 7.35. When $F = \mathbb{Q}$ we get $\tau_n \in \mathbb{Z}[T]$. Such Chebyshev polynomials arise from the trigonometric identity $2\cos(nT) = \tau_n(2\cos(T))$.

The following is well-known, and the second part is easily verified.

Lemma 7.36. *Let n be an odd prime which is prime to the characteristic of F. Then the nth Chebyshev polynomial $\tau_n \in F[T]$ has a dihederal geometric monodromy group of order $2n$, and this group acts on $\{x_i\}$ via the standard dihederal action.*

In particular, $\tau_n(T) - \tau_n(S)$ has $<\chi_{\tau_n}, \chi_{\tau_n}> = (n+1)/2$ irreducible factors. All are quadratic, except for the linear factor $T - S$. So the action of the geometric monodromy group on $\{x_i\}$ is doubly transitive only for $n = 3$.

The following result of Burnside is an important piece in the classification.

Lemma 7.37. *Suppose G acts effectively and transitively, but not doubly transitivly, on a set $\mathcal{S}$ of prime order l. Then G is isomorphic to a subgroup of the affine group $\mathbb{F}_l \rtimes \mathbb{F}_l^\times$.*

The last piece is provided by the following.

Lemma 7.38. *Let $f \in F[T]$ be a polynomial whose degree is a prime distinct from the characteristic of F. If the geometric monodromy group G_f is solvable, then f is linearly related, over $\overline{F}$, to either a cyclic polynomial or a Chebyshev polynomial.*

Proof. This was first proved for the case $f : \mathbb{P}^1 \to \mathbb{P}^1$ a tame cover ([Fri70]). For the general case see [Mül97]. Here we give the barest sketch of the proof.

If G_f is solvable, then by analyzing and comparing the possible ramification associated to $\Omega_f/F(x_i)$ and $\Omega_f/F(z)$, and by using the Riemann-Hurwitz formula, conclude that the cover $f : \mathbb{P}^1 \to \mathbb{P}^1$ is totally ramified above one or two points. If it is totally ramified above two points, then G_f is cyclic, and a simple affine change of coordinates shows f is linearly related to the cyclic polynomial T^n. If the cover is totally ramified above only one point, G_f is dihedral of order $2n$, and f can be shown to be linearly related to a Chebyshev polynomial. □

Putting all this together gives the following classification.

Proposition 7.39. *Suppose $f \in F[T]$ has degree prime to the characteristic of F. Then the geometric monodromy group acts doubly transitively on the roots $\{x_i\}$ if and only if one of the following hold.*

(7.14) *f is indecomposable of composite degree.*
(7.15) *f has degree 2.*
(7.16) *f has degree 3 and is not linearly related to the cyclic polynomial T^3.*
(7.17) *f has prime degree $n > 3$ and is not linearly related over $\overline{F}$ to either the cyclic polynomial or the Chebyshev polynomial of degree n.*

Remark 7.40. Suppose that $f \in F[T]$ has degree prime to the characteristic of F. It is easy to show that if f is linearly related over $\overline{F}$ to a cyclic polynomial, then f is linearly related over F to a cyclic polynomial.

For Chebychev polynomials, the situation is more complicated. For any $a \in F$, and positive integer n define the *Dickson Polynomial* $D_n(a, T)$ to be the cyclic polynomial if $a = 0$, or $D_n(a, T) = a^{n/2}\tau_n(a^{-1/2}T)$ if $a \neq 0$. One can show that $D_n(a, T)$ is in $F[T]$ for all $a \in F$. Observe that any non-cyclic Dickson polynomial is linearly related over a quadratic extension of F to a Chebyshev polynomial.

From [Turn95, Lemma 1.9], it follows that if f is linearly equivalent over $\overline{F}$ to a Chebyshev polynomial, then it is linearly equivalent over F to a Dickson polynomial.

7.4. **A Special Class of Davenport Pairs.** Recall that one way to construct a DP (f, g) is to set $f = f' \circ h_1$ and $g = g' \circ h_2$ where (f', g') is a SDP and $h_1, h_2 \in \mathbb{F}_q[T]$ are exceptional. Such DP's have the property that $1 \in \mathcal{D}_{f,g}$.

How does one construct DP's (f, g) with $1 \notin \mathcal{D}_{f,g}$? One strategy is to consider $f, g \in \mathbb{F}_q[T]$ with $g = f \circ l$ for some linear polynomial $l \in \overline{\mathbb{F}}_q[T]$ *not* in $\mathbb{F}_q[T]$. We show that the only examples of this type, when f is indecomposable of degree prime to the characteristic of $\mathbb{F}_q$, are essentially of the form (T^n, aT^n) where $a \in \mathbb{F}_q$ is not an nth power in $\mathbb{F}_q$ (Corollary 7.43); in this case $f(T) = T^n$ and $l(T) = a^{1/n}T$.

Lemma 7.41. *Let $f \in F[T]$ be linearly related over $\overline{F}$ to a Chebyshev polynomial of odd degree prime to the characteristic of F. Suppose that $f(\alpha T + \beta) \in F[T]$ for some $\alpha, \beta \in \overline{F}$, $\alpha \neq 0$. Then $\alpha, \beta \in F$.*

Proof. This follows from [Turn95, Lemma 1.9]. (Our $\tau_n(T)$ is equal to Turnwald's $D_n(1, T)$). □

Proposition 7.42. *Let $f, g \in F[T]$ be indecomposable polynomials of degree n prime to the characteristic of F. Suppose F is a perfect field. If f and g are linearly related on the inside over $\overline{F}$ then either (i) f and g are linearly related on the inside over F, or (ii) f and g are both linearly related over F to the cyclic polynomial of degree n. In either case, f and g are linearly related over F.*

Proof. Observe that $n < 3$ is trivial. If $G_{f,g}$ acts doubly transitively on $\{x_i\}$, use Lemma 7.28. If $G_{f,g}$ does not act doubly transitively, use Proposition 7.39 to reduce to the Chebyshev or cyclic case. In the case where f and g are linearly related over $\overline{F}$ to the Chebyshev polynomial and n is an odd prime, use the previous lemma. Finally, in the cyclic case, Remark 7.40 says that f and g are linearly related to T^n over the base field F. □

Corollary 7.43. *Suppose that $f \in F[T]$ is indecomposable of degree n prime to the characteristic of F where F is a perfect field. If $g \in F[T]$ is linearly related to f on the inside over $\overline{F}$, but not over F, then there are linear $l_1, l_2, l_3 \in F[T]$ such that $l_1 \circ f \circ l_2 = T^n$ and $l_1 \circ g \circ l_3 = aT^n$ with $a \in F$ not an nth power in F.*

8. Main Results Concerning Indecomposability

The following results concern the case where one of the polynomials (f, say) of the pair (f, g) is indecomposable with degree prime to the characteristic. There are essentially two cases, depending on whether or not f is linearly related to a cyclic polynomial. Recall f is linearly related to a cyclic polynomial over $\mathbb{F}_q$ if and only if it is linearly related to a cyclic polynomial over $\overline{\mathbb{F}}_q$.

Theorem 8.1. *Let $f \in \mathbb{F}_q[T]$ be indecomposable over $\mathbb{F}_q$, non-exceptional, and of degree prime to the charactersitic of $\mathbb{F}_q$. Let $g \in \mathbb{F}_q[T]$ be any polynomial where (f, g) forms a Davenport Pair, and let $g = g_1 \circ g_2$ be the induced decomposition over $\mathbb{F}_q$ associated to (f, g).*

If f is not linearly related to a cyclic polynomial then (f, g_1) is a SDP. In fact, it is a SDP with multiplicity: the associated characters χ_f, χ_{g_1} are equal.

If f is linearly related to a cyclic polynomial, then $g = f \circ h$ for some $h \in E[T]$ where E is a finite extension of $\mathbb{F}_q$. Furthermore, $g = l \circ f \circ h'$ for some $l, h' \in \mathbb{F}_q[T]$ with l linear, and f and $l \circ f$ are linearly related on the inside over $\overline{\mathbb{F}}_q$.

Remark 8.2. Throughout this paper we have adopted the convention that polynomials have non-zero derivatives. However, the above theorem (and the following theorem) remain valid for g with zero derivative (with a suitable definition of *induced decomposition*).

Proof. Let $g = h_1 \circ h_2$ be the induced decomposition over $\overline{\mathbb{F}}_q$ associated with the pair (f, g). (It turns out, at least in the non-cyclic cases, that the two induced decompositions, $g = g_1 \circ g_2$ and $g = h_1 \circ h_2$, are equivalent).

Since (f, g) is a DP and f is non-exceptional, $f(T) - g(S)$ is reducible in $\overline{\mathbb{F}}_q[S, T]$ (Corollary 4.12). So $f(T) - h_1(S)$ is also reducible in $\overline{\mathbb{F}}_q[S, T]$ (Lemma 6.2). Since f is indecomposable over $\mathbb{F}_q$, f is indecomposable over $\overline{\mathbb{F}}_q$ (Theorem 3.5 of [FM69]). Thus the induced decompositions of both f and h_1, associated to the pair (f, h_1) over $\overline{\mathbb{F}}_q$, are trivial. Lemma 6.1, especially property (6.2), implies $\overline{\mathbb{F}}_q \Omega_f = \overline{\mathbb{F}}_q \Omega_{h_1}$ and $G_f = G_{h_1} = G_{f,h_1}$. Finally, $\deg f = \deg h_1$ (Corollary 6.3).

Now we show, under the assumption that $f(T) - g(S)$ is reducible over $\mathbb{F}_q$, that we can take h_i to be g_i for $i = 1, 2$. Note: the above argument that shows

$\deg f = \deg h_1$ can be modified to show that $\deg f = \deg g_1$ under this reducibility assumption. Now, by Lemma 6.1,

$$\mathbb{F}_q(y_1) \cap \Omega_f = \mathbb{F}_q(g_2(y_1)) \quad \text{and} \quad \overline{\mathbb{F}}_q(y_1) \cap (\overline{\mathbb{F}}_q\Omega_f) = \overline{\mathbb{F}}_q(h_2(y_1)).$$

Since $\overline{\mathbb{F}}_q(\mathbb{F}_q(y_1) \cap \Omega_f) \subseteq \overline{\mathbb{F}}_q(y_1) \cap (\overline{\mathbb{F}}_q\Omega_f)$,

$$\overline{\mathbb{F}}_q(g_2(y_1)) \subseteq \overline{\mathbb{F}}_q(h_2(y_1)).$$

In particular, $g_2 = h' \circ h_2$ for some polynomial $h' \in \overline{\mathbb{F}}_q$. Observe that $\deg h' = 1$ since $\deg h_1 = \deg g_1$, so, after adjusting h_1 and h_2 by a linear map, $h_i = g_i$, for $i = 1, 2$.

We divide the remainder of the proof into three cases, using Proposition 7.39.

Case 1: *G_f acts doubly transitively on the roots $\{x_i\}$ and* $\deg f > 2$. By Proposition 7.26 and Corollary 7.30, $f(T) - h_1(S)$ has exactly two irreducible factors over $\overline{\mathbb{F}}_q$, and these factors have non-equal degrees. By Lemma 6.2, the factorization of $f(T) - g(S)$ is recovered by substituting $h_2(S)$ for S in the factorization of $f(T) - h_1(S)$. So the two irreducible factors of $f(T) - g(S)$ have non-equal T-degrees, and must then be fixed under the action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Thus the factorization of $f(T) - g(S)$ is defined over $\mathbb{F}_q$. As discussed above, this means we can take $h_1 = g_1$. The result follows from Proposition 7.26 and Corollary 7.27.

Case 2: *f is linearly related over $\overline{\mathbb{F}}_q$ to a Chebyshev polynomial and $n = \deg f$ is an odd prime.* Let $G = G_f = G_{h_1}$. By Lemma 7.36, G is isomorphic to a dihedral group of order $2n$. Note: G acts transitively on both $\{x_i\}$ and on the roots $\{u_j\}$ of $h_1(T) - z$. Clearly, any two transitive actions of such a dihederal group on sets of order n are equivalent as permutation representations. Thus $G(x_1)$ is $G(u_j)$ for some j. Use the description of factorization of Remark 4.9 applied to $G(u_j) = G(x_1)$ acting on $\{x_i\}$, to conclude that the factorization of $f(T) - h_1(S)$ has exactly one linear factor Φ and $(n-1)/2$ irreducible quadratic factors in $\overline{\mathbb{F}}_q[S, T]$.

The factorization of $f(T) - g(S)$ can be recovered by substituting $h_2(S)$ for S in the factorization of $f(T) - h_1(S)$ (Lemma 6.2). Since $\Phi(T, h_2(S))$ is the unique irreducible factor of $f(T) - g(S)$ of T-degree one, it is fixed under the action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. So $f(T) - g(S)$ is reducible in $\mathbb{F}_q[S, T]$. As discussed above, we can conclude that $h_1 = g_1$. Also, Φ, the only linear factor of $f(T) - g_1(S)$, must be defined over $\mathbb{F}_q$. The existence of Φ implies that f and g_1 are linearly related on the inside over $\mathbb{F}_q$. Thus (f, g_1) forms a trivial SDP.

Case 3: *f is linearly related over $\overline{\mathbb{F}}_q$ to a cyclic polynomial.* (This automatically includes the case $\deg f = 2$). Let $G = G_f = G_{h_1}$. So G is isomorphic to a cyclic group of order n acting transitively on both $\{x_i\}$ and on roots $\{u_j\}$ of $h_1(T) - z$. Clearly, any two transitive actions of G on sets of order n are equivalent as permutation representations. Thus $G(x_1)$ is $G(u_j)$ for some j. Use Remark 4.9 to conclude that $f(T) - h_1(S)$ factors over $\overline{\mathbb{F}}_q$ as the product of n linear factors. This implies that $h_1 = f \circ l_0$ for some $l_0 \in \overline{\mathbb{F}}_q[T]$ of degree 1. So $g = f \circ h$ where $h = l_0 \circ h_2$.

Using [FGS93, Lemma 4.1] and the fact that $\deg f$ is prime to p, we get a linear polynomial $l' \in \overline{\mathbb{F}}_q[T]$ such that $f' = f \circ l'$ and $h' = (l')^{-1} \circ h$ are in $\mathbb{F}_q[T]$, giving a decomposition $g = f' \circ h'$ over $\mathbb{F}_q$. By Proposition 7.42, $f' = l \circ f \circ l''$ for some linear $l, l'' \in \mathbb{F}_q[T]$. By replacing h' with $l'' \circ h'$, we obtain the decomposition $g = l \circ f \circ h'$. □

Remark 8.3. We can get a variant of the above theorem by replacing the hypotheses *(f, g) is a DP and f is not exceptional* with the alternate hypothesis *$f(T) - g(S)$ reducible over $\overline{\mathbb{F}}_q$* (keeping all the other hypotheses as they are).

Theorem 8.4. *Let $f, g \in \mathbb{F}_q[T]$ be two polynomials such that $\mathcal{V}_g(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_f(\mathbb{F}_{q^t})$ for all t. Suppose that f is indecomposable over $\mathbb{F}_q$ and has degree prime to the characteristic of $\mathbb{F}_q$. Then there are polynomials $g_1, g_2 \in \mathbb{F}_q[T]$ such that $g = g_1 \circ g_2$ and (f, g_1) is a SDP with multiplicity.*

Proof. Consider Remark 8.3 together with Proposition 4.16. The case where f is not linearly related to a cyclic polynomial follows immediately.

In the cyclic case, consider the decomposition $g = l \circ f \circ h'$ of Theorem 8.1, and let $f' = l \circ f$. We know that f and f' are linearly related on the inside over $\overline{\mathbb{F}}_q$; we claim that they are in fact linearly related on the inside over $\mathbb{F}_q$, and so we can take $g_1 = f$. Suppose otherwise and use Corollary 7.43 to reduce to the case $f = T^n$ and $f' = aT^n$ where $a \in \mathbb{F}_q$ is not an nth power. Let t be such that $q^t > \deg h'$ but that a is not an nth power in $\mathbb{F}_{q^t}$. Then $\mathcal{V}_f(\mathbb{F}_{q^t})$ contains only nth powers, but if $c \in \mathbb{F}_{q^t}$ not a root of h' then $g(c)$ is not an nth power, a contradiction. □

Remark 8.5. In the above theorems, there are many cases where we can conclude that (f, g_1) is actually a trivial SDP. In other words, we can choose the decomposition $g = g_1 \circ g_2$ in such a way that $g_1 = f$ itself.

For example, in case 2 of the above proof we concluded that (f, g_1) is a trivial SDP if $n = \deg f$ is an odd prime, and f is linearly related to a Chebyshev polynomial. In this case G_f is dihedral. In fact, there is no hope of having (f, g_1) be a non-trivial SDP unless G_f has two non-equivalent permutation representations on a set of n elements whose associated characters are equal (this follows from Proposition 7.42). This excludes most G_f.

The classification of finite simple groups, in the form of the classification of doubly transitive representations ([CKS76]), can be used to classify groups G with two non-equivalent faithful permutation representations acting on a set with n elements such that (i) the characters of the two actions are equal, (ii) the actions are doubly transitive, (iii) there is an element of G which acts as an n-cycle under the two actions. The conclusion is that

$$G = \mathrm{PSL}_2(\mathbb{F}_{11}) \quad \text{and} \quad n = 11,$$

or

$$\mathrm{PSL}_k(\mathbb{F}_s) \subseteq G \subseteq \mathrm{P\Gamma L}_k(\mathbb{F}_s) \quad \text{and} \quad n = (s^k - 1)/(s - 1) \text{ for some } k \geq 3.$$

(This result was conjectured in [Fri73]; see Theorem 2.7 and Section 9 of [Fri99] for more details, including historical information.) The field $\mathbb{F}_s$ appearing in the above list is called the *characteristic field* of the Chevalley group G.

This result allows us to strengthen the above theorems: if $G = G_f$ and n are not of the above form, then the conclusion *(f, g_1) is a SDP*, can be replaced by the stronger conclusion *$g = f \circ h$ for some $h \in \mathbb{F}_q[T]$*.

However, not all the above groups are expected to occur as geometric monodromy groups of polynomials (for a given $\mathbb{F}_q$). In fact, Guralnick has conjectured the following: the finite simple groups appearing as composition factors of geometric monodromy groups G_f as f varies over all polynomials, or even all rational functions, are, with finitely many exceptions (depending on the characteristic), the cyclic groups, the alternating groups, and the Chevalley groups with characteristic field containing $\mathbb{F}_p$. Thus, we can expect among $f \in \mathbb{F}_q[T]$ with $\mathbb{F}_q$ of fixed characteristic p, that the fields $\mathbb{F}_s$ appearing as G_f as in the above classification should, with a finite number of exceptions depending on p, also be of characteristic p.

By way of contrast, in the case where $\mathbb{F}_q$ and $\mathbb{F}_s$ have the same characteristic, examples abound. Theorem 5.2 of [Fri99] (dependent on [Abh97]) states that, for any finite field $\mathbb{F}_q$, any s a power of the characteristic of $\mathbb{F}_q$, and any $k \geq 3$, there is a non-trivial SDP (f, g) with $\chi_f = \chi_g$ whose geometric monodromy group is $G_f = G_g = \mathrm{PSL}_k(\mathbb{F}_s)$.

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